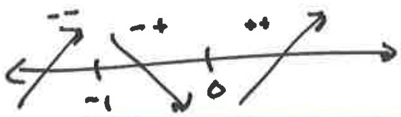
Math 90 2^{nd} Midterm Solutions

Section 01 Tom Hulse + David Lowry
(written by David Lowry)

1 $f(x) = 2x^3 + 3x^2$

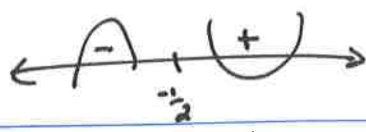
(A) $f'(x) = 6x^2 + 6x = 6x(x+1)$

$f'(x) = 0 \Longrightarrow 6x(x+1) = 0 \longrightarrow x=0,\ x=-1$



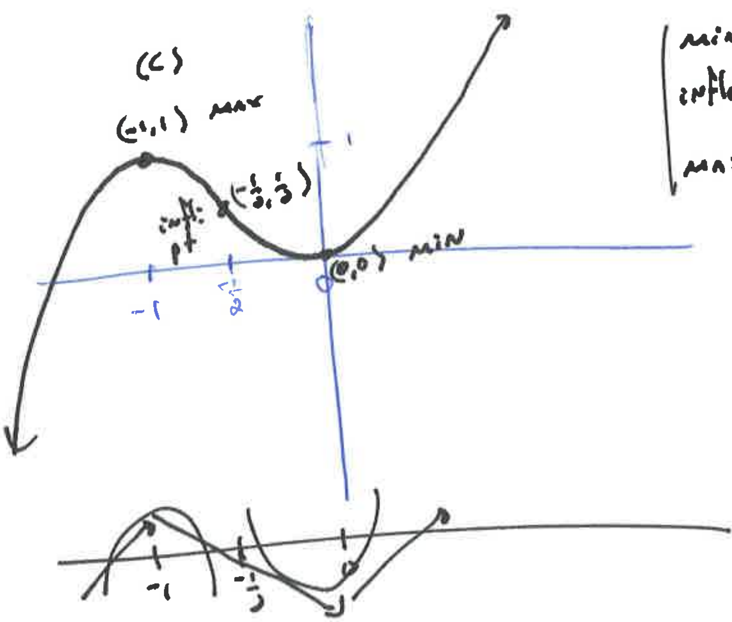
$\boxed{\text{increasing on } (-\infty,-1),\ (0,\infty) \quad \text{decr: on } (-1,0)}$

(b) $f''(x) = 12x + 6$, $\quad f''(x) = 0 \Longrightarrow 12x = -6 \Rightarrow x = -\frac{1}{2}$



$\boxed{\text{concave down}: (-\infty, -\tfrac{1}{2}),\quad \text{concave up } (-\tfrac{1}{2}, \infty)}$

(c)



min: $f(0) = 0$

inflection: $f(-\frac{1}{2}) = -\frac{2}{8} + \frac{3}{4} = \frac{1}{2}$

max: $f(-1) = -2 + 3 = 1$

$\blacksquare$

2 (a) $\displaystyle\lim_{x\to 1} \frac{x \ln x}{\sin(\pi x)}$ $\qquad$ $\begin{matrix}\ln(1) = 0 \\ \sin(\pi) = 0\end{matrix} \Longrightarrow$ looks like $\frac{0}{0}$

$\Longrightarrow$ use l'Hôpital's rule

$$\lim_{x\to 1} \frac{x \ln x}{\sin(\pi x)} = \lim_{x\to 1} \frac{\frac{x}{x} + \ln x}{\pi \cos(\pi x)} = \frac{1 + 0}{\pi(-1)} = \boxed{-\frac{1}{\pi}}$$

(b) $\displaystyle\lim_{x\to 0} \frac{\tan^{-1}(2x)}{x^3 + 3x} = \cdots$ $\qquad$ $\begin{matrix}\tan^{-1}(0) = 0 \\ x^3 + 3x\Big|_{x=0} = 0\end{matrix} \longrightarrow$ looks like $\frac{0}{0}$

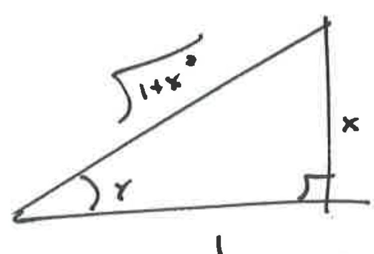
$\longrightarrow$ use l'Hôpital's rule

$$\lim_{x\to 0} \frac{\tan^{-1}(2x)}{x^3 + 3x} = \lim_{x\to 0} \frac{\frac{2}{1+4x^2}}{3x^2 + 3} = \frac{\frac{2}{1+0}}{0+3} = \boxed{\frac{2}{3}}$$

Aside: suppose you forget $\frac{d}{dx} \tan^{-1}(x)$. What do you do?
First, you skip this question + return to it after trying the rest.
Then: $\quad y = \tan^{-1}(x) \longrightarrow \tan y = x$

$\qquad$ so $\frac{d}{dx} \tan y = \frac{d}{dx} x = 1$

$\qquad$ + $\frac{d}{dx} \tan y = \sec^2 y \cdot \frac{dy}{dx}$



a triangle w/ $\tan y = x$.

From triangle at right, $\sec^2 y = 1 + x^2$.
And $y = \tan^{-1} x \Longrightarrow \frac{dy}{dx} = \frac{d}{dx} \tan^{-1} x$.
Putting it all together, $\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+x^2}$ //

$\boxed{3}$ (I wrote this problem)

x kilograms of powder, $\longrightarrow$ $100x - x^2$ cartons of chalk

\$1000 overhead, \$5/carton profit, \$30/kilo powder.

(a) $P(x) = (100x - x^2 \text{ cartons}) \frac{\$5}{\text{carton}} - (x \text{ kilos powder}) \frac{\$30}{\text{kilo}} - \$1000$

$= 5(100x - x^2) - 30x - 1000$

$\boxed{= 470x - 5x^2 - 1000}$

(b) $P'(x) = 470 - 10x$

$P'(x) = 0 \implies 470 = 10x \longrightarrow x = 47.$

Is it the MAX?

$P''(x) = -10 < 0 \implies$ it is a max by 2nd deriv. test. Since we're letting x be any number, $P(x)$ is differentiable everywhere, + we've tested all critical points, $x = 47$ is the MAX.

So the company should buy 47 kilograms of chalk.

Note $\longrightarrow$ you do need to check that it's A MAX.

$\boxed{4}$ $f(x)$ continuous, s.t. $f''(x) = x^3 + e^x + 1$, $f'(2) = 0$.

(a) $f'(x) = \dfrac{x^4}{4} + e^x + x + c$

$f'(2) = 4 + e^2 + 2 + c = 0 \implies c = -e^2 - 6$

$$\implies f'(x) = \boxed{\frac{x^4}{4} + e^x + x + (-e^2 - 6).}$$

(b) $f(x) = \dfrac{x^5}{20} + e^x + \dfrac{x^2}{2} + x(-e^2-6) + c_2.$

so $f(x)$ could be, say,

$$\boxed{\begin{array}{l} \dfrac{x^5}{20} + e^x + \dfrac{x^2}{2} + x(-e^2-6) + 2, \text{ or} \\ \dfrac{x^5}{20} + e^x + \dfrac{x^2}{2} + x(-e^2-6) - 1 \end{array}}$$ (or many others).

(c) We know $f'(2) = 0$ by design, so $x = 2$ will be the location of the extrema.

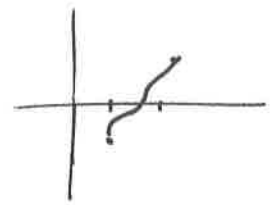
$f''(x) = x^3 + e^x + 1$, $f''(2) = 8 + e^2 + 1 > 0$,

So by 2nd deriv. test, $f(x)$ has a

$\boxed{\text{\underline{MINIMUM} at } x=2.}$

Note: you can check your answers: differentiate f to get f', + f' to get f''. //

$\boxed{5}$ $h(x) = 4x + 2\ln x - 5$



(A) $h(1) = 4 + 0 - 5 = -1 < 0$

$h(2) = 8 + 2\ln 2 - 5 > 0$

By intermediate value theorem, h has a zero in $[1,2]$ since h is continuous. So there is at <u>least one</u> <u>root.</u>

(b) $h'(x) = 4 + \frac{1}{x}$

+ $h'(x) > 0$ for $x > 0$, thus $h(x)$ is always strictly increasing, + thus <u>has exactly one root.</u>

$\blacksquare$

$\boxed{6}$ $f(x) = e^{-2x}(x^2-4x+2)$

(a) $f'(x) = -2e^{-2x}(x^2-4x+2) + e^{-2x}(2x-4) =$

$= e^{-2x}\left(-2x^2+8x-4+2x-4\right) = e^{-2x}(-2x^2+10x-8) =$

$= -2e^{-2x}\left(x^2-5x+4\right) = -2e^{-2x}(x-4)(x-1)$

$\Longrightarrow$ critical points are $\boxed{x=1,4}$.

(b) max/min on $[0,3]$

$\longrightarrow$ 3 candidate points: $x=0$, $x=1$, $x=3$ (not $x=4$).

try them all: $f(0) = e^0(0-0+2) = 2$

$f(1) = e^{-2}(1-4+2) = -e^{-2}$

~~$f(2) =$~~

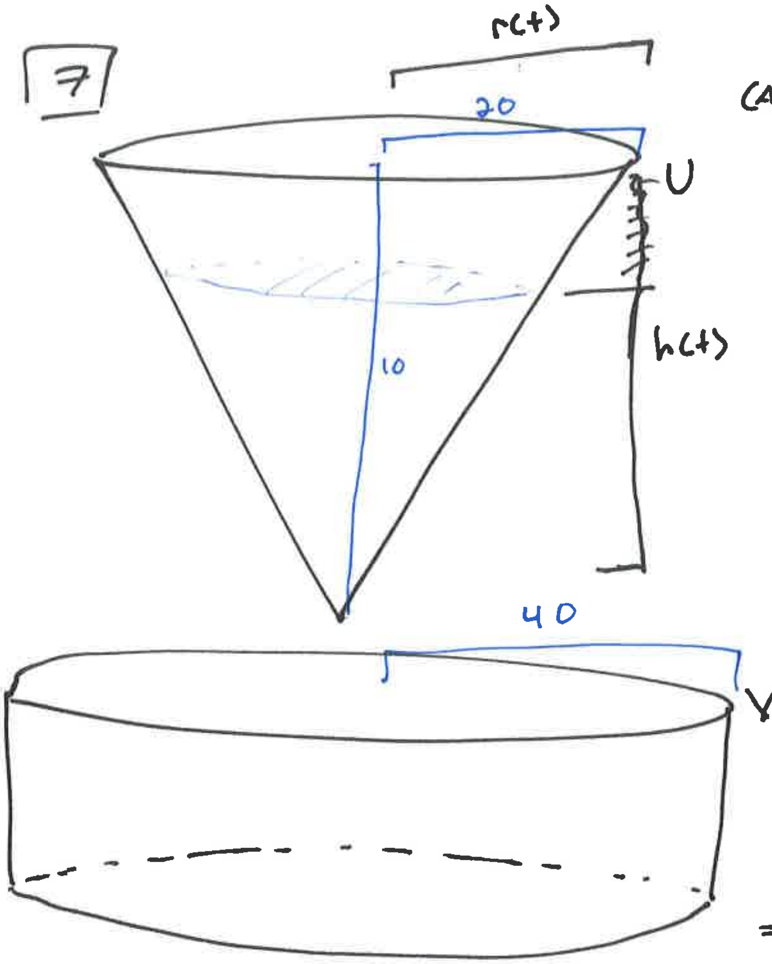
$f(3) = e^{-6}(9-12+2) = -e^{-6}$

so the max is at $x=0$, as $2 > -e^{-6} > -e^{-2}$

+ the min is at $x=1$, as $2 > -e^{-6} > -e^{-2}$.

$$\boxed{\begin{array}{l} \text{max: } 2 \\ \text{min: } -e^{-2} \end{array}}$$

7



(a) $U(t) = \frac{\pi}{3} r(t)^2 h(t)$

it's a cone, so the radius + height of water at time t are always in a fixed proportion.

Here, $r(t) = 2h(t)$.

So $U(t) = \frac{\pi}{3}(2h(t))^2 h(t)$

$= \frac{4\pi}{3} h(t)^3$

$\rightarrow U'(t) = 4\pi h(t)^2 \frac{dh}{dt}$

$\Rightarrow$ when $h(t) = 5$, $U'(t) = 4\pi(25)\frac{dh}{dt}$

+ $\frac{dh}{dt} = -6$ (negative since it's dropping)

$\Rightarrow U' = 4\pi(25)\cdot -6 = \boxed{-600\pi}$ m^3/min.

(b) Water falls out of the cone + into the cylinder.

So $\boxed{\frac{dU}{dt} = -\frac{dL}{dt}.}$

$\boxed{7}$ continued

(c) $L(t) = \pi (40)^2 h_L(t)$

$$\Longrightarrow \frac{dL}{dt} = 1600\pi \frac{dh_L}{dt}$$

$$\Longrightarrow 600\pi = 1600\pi \frac{dh_L}{dt}$$

$$\Longrightarrow \frac{6\cancel{00}\,\cancel{\pi}}{16\cancel{00}\,\cancel{\pi}} = \frac{3}{8} = \frac{dh_L}{dt}$$

so it's rising at $\boxed{\frac{3}{8} \text{ m/min.}}$

$\blacksquare$