A BIT MORE ABOUT PARTIAL FRACTION DECOMPOSITION

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This is a short note written for my students in Math 170, talking about partial fraction decomposition and some potentially confusing topics that have come up. We'll remind ourselves what partial fraction decomposition is, and unlike the text, we'll prove it. Finally, we'll look at some pitfalls in particular. All this after the fold.

Contents

1. The Result Itself

We are interested in *rational functions* and their integrals. Recall that a polynomial $f(x)$ is a function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where the a_i are constants and x is our "intederminate" – and which we commonly imagine standing for a number (but this is not necessary).

Then a rational function $R(x)$ is a ratio of two polynomials $p(x)$ and $q(x)$,

$$R(x) = \frac{p(x)}{q(x)}.$$

Then the big result concerning partial fractions is the following:

Theorem 1.1. *If $R(x) = \dfrac{p(x)}{q(x)}$ is a rational function and the degree of $p(x)$ is less than the degree of $q(x)$, and if $q(x)$ factors into*

$$q(x) = (x-r_1)^{k_1}(x-r_2)^{k_2}\ldots(x-r_l)^{k_l}(x^2+a_{1,1}x+a_{1,2})^{v_1}\ldots(x^2+a_{m,1}x+a_{m,2})^{v_m},$$

then $R(x)$ can be written as a sum of fractions of the form $\dfrac{A}{(x-r)^k}$ or $\dfrac{Ax+B}{(x^2+a_1 x+a_2)^v}$, where in particular

- If $(x - r)$ appears in the denominator of $R(x)$, then there is a term $\dfrac{A}{x - r}$
- If $(x - r)^k$ appears in the denominator of $R(x)$, then there is a collection of terms
$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_k}{(x - r)^k}$$
- If $x^2 + ax + b$ appears in the denominator of $R(x)$, then there is a term $\dfrac{Ax + B}{x^2 + ax + b}$
- If $(x^2 + ax + b)^v$ appears in the denominator of $R(x)$, then there is a collection of terms
$$\frac{A_1 x + B_1}{x^2 + ax + b} + \frac{A_2 x + B_2}{(x^2 + ax + b)^2} + \cdots \frac{A_v x + B_v}{(x^2 + ax + b)^v}$$

where in each of these, the capital A and B represent some constants that can be solved for through basic algebra.

I state this result this way because it is the one that leads to integrals that we can evaluate. But in principle, this theorem can be restated in a couple different ways.

Let's parse this theorem through an example - the classic example.

Example 1.2. *Consider the rational function $\frac{1}{x(x+1)^2}$. The terms that appear in the denominator are x and $(x + 1)^2$. The x part contributes an $\dfrac{A}{x}$ term. The $(x + 1)^2$ part contributes a $\dfrac{B}{x + 1} + \dfrac{C}{(x + 1)^2}$ pair of terms. So we know that*
$$\frac{1}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2},$$
and we want to find out what A, B, C are. Clearing denominators yields
$$1 = A(x + 1)^2 + Bx(x + 1) + Cx = (A + B)x^2 + (2A + B + C)x + A,$$
and comparing coefficients of the polynomial 1 and $(A + B)x^2 + (2A + B + C)x + A$ gives immediately that $A = 1, B = -1, and C = -1$. So
$$\frac{1}{x(x + 1)^2} = \frac{1}{x} + \frac{-1}{x + 1} + \frac{-1}{(x + 1)^2}.$$
It is easy (and recommended!) to check these by adding up the terms on the right and making sure you get the term on the left. □

2. Common Pitfalls

Very often in math classes, students are "lied to" in one of two ways: either results are stated that are far weaker than normal, or things are said about the impossibility to do something... when it's actually possible. For example, middle school teachers might often say that taking the square root

of negative numbers "isn't allowed" or "doesn't mean anything," when really there is a several hundred year tradition of doing just that. (On the other hand, things are much more complicated in some ways once we allow $\sqrt{-1}$, so it makes sense to defer its treatment).

Perhaps because of this, students often try to generalize the statement of partial fractions, which applies to *rational* functions, to other types of functions. But it is *very important* to remember that partial functions works for rational functions, i.e. for ratios of polynomials. So if you have $\dfrac{1}{x\sqrt{x-1}}$, you cannot naively apply the partial fractions algorithm, as $x\sqrt{x-1}$ is not a polynomial.

As an aside, we can be a bit clever. If you call $y = \sqrt{x-1}$, so that $y^2 + 1 = x$, then we see that $\dfrac{1}{x\sqrt{x-1}} = \dfrac{1}{y(y^2+1)}$, which you *can* approach with partial fractions. You should check that

$$\frac{1}{y(y^2+1)} = \frac{1}{y} - \frac{y}{y^2+1},$$

so that

$$\frac{1}{x\sqrt{x-1}} = \frac{1}{\sqrt{x-1}} - \frac{\sqrt{x-1}}{x}.$$

So while *something* is possible here, it's not a naive application of partial fractions.

Similarly, if you have something like $\dfrac{\sin\theta}{\cos^2\theta + \cos\theta}$, you cannot apply partial fractions because you are not looking at a rational function.

There's another common danger, which has to do with what you assume is true. For example, if you assume that you *can* use partial fractions on $\dfrac{1}{x\sqrt{x-1}}$ (which you cannot!), then you might do something like

$$\frac{1}{x\sqrt{x-1}} = \frac{A}{x} + \frac{B}{\sqrt{x-1}}, \tag{2.1}$$

so that clearing denominators gives

$$1 = A\sqrt{x-1} + Bx$$

You might then thing that setting $x = 1$ shows that $B = 1$, and setting $x = 10$ gives $3A + B = 3A + 1 + 1$, meaning that $A = 0$. And so $Bx = 1$. But this is clearly nonsense. And the issue here is that the initial equation 2.1 is not true - starting with faulty assumptions gets you no where.

A key thing to remember is that you can always check your work by just adding together the final decomposition after finding a common denominator! And if you have a good feel for functions, you should be able to realize that no linear combination of $\sqrt{x-1}$ and x will ever be the constant 1 - so that final equality will never be possible.

3. A Proof (more or less)

Giving the proof for the repeated factor part is annoying, but very similar to the non-repeated root case. Suppose that we have a number r and a polynomial $q(x)$ such that $q(r) \neq 0$. Under these assumptions, we will show that there is a polynomial $p(x)$ of degree less than $q(x)$ and a number A such that

$$\frac{1}{q(x)(x-r)} = \frac{p(x)}{q(x)} + \frac{A}{x-r}.$$

This is clearly equivalent to finding a polynomial $p(x)$ and A such that

$$1 = p(x)(x-r) + Aq(x)$$

We want this as an equality of polynomials, meaning it holds for all x. So in particular, it should hold when $x = r$, leading us to the equality

$$1 = Aq(r),$$

which can be rewritten as

$$A = \frac{1}{q(r)}$$

as $q(r) \neq 0$. So we have found A.

We are left with $p(x)(x-r) = 1 - Aq(x)$. By our choice of A, we see that the right hand side is 0 when $x = r$, so that the right hand side has $x - r$ as a factor. So $1 - Aq(x) = N(x)(x-r)$ for some polynomial N of degree smaller than the degree of $q(x)$. (We have used the Factor Theorem here, which says that if a is a root of $p(x)$, then $p(x) = p_1(x)(x-a)$ for a smaller degree polynomial $p_1(x)$). Choosing $p(x)$ to be this $N(x)$ gives us this equality as well, so that we have found a satisfactory A and $p(x)$.

This lets us peel off the (non-repeating) factors of the denominator one at a time, one after the other, to prove the theorem for cases without repeated roots. The case with repeated roots is essentially the exact same, and would be a reasonable thing to try to prove on your own. (Hint: there will be a point when you might want to divide everything by $x - r$).

4. Conclusion

So that's that about partial fractions. If there are any questions, feel free to let me know. This post was typeset in the LaTeX typeset language, hosted on a Wordpress site davidlowryduda.com, and displayed there with MathJax. This can also be found in pdf note form, and the conversion from note to Wordpress is done using a customized version of latex2wp that I call mse2wp, located at github.com/davidlowryduda/mse2wp.

Thank you, and I'll see you in class.