Homework #8 Solutions

$\boxed{35.3}$

(a) $(3-2i)(1+4i) = 3 - 2i + 12i - 8i^2$

$= 11 + 10i$

(b) $\dfrac{3-2i}{1+4i} = \dfrac{3-2i}{1+4i}\dfrac{(1-4i)}{(1-4i)} = \dfrac{-5-14i}{9} = -\dfrac{5}{9} - \dfrac{14}{9}i$

(c) $\left(\dfrac{1+i}{\sqrt{2}}\right)^2 = \frac{1}{2}(1+i)^2 = \frac{1}{2}(1+2i+i^2) = i.$

[Is that surprising? For those that know the pictoral representation of complex multiplication, this might feel natural.] ∎

$\boxed{35.4}$

$x = u+iv$. Then $x^2 = (u+iv)^2 = u^2 - v^2 + 2uvi$.

We want $u^2 - v^2 + 2uvi = 95 - 168i$,

or
$$u^2 - v^2 = 95$$
$$2uv = -168.$$

By running through the factorizations of $\frac{-168}{2} = uv$, or by setting $u = \frac{-168}{2v}$ & using the quadratic formula, we solve for u and v and get

$$(u,v) = (-12, 7) \text{ or } (12, -7).$$

So $x = -12 + 7i$ or $x = 12 - 7i$. ∎

For $x^2 = 1+2i$, we need to solve $\begin{cases} u^2 - v^2 = 1 \\ 2uv = 2 \end{cases}$. But the second line has $(u,v) = (1,1)$ or $(-1,-1)$ as the only solutions, & neither work. So there are no solutions there. ∎

$\boxed{\#3}$ We define $N(z) = a^2+b^2$ for $z = a+bi$.

Then $z \cdot \bar{z} = (a+bi)(a-bi) = a^2 + abi - abi - b^2 i^2$
$= a^2+b^2$,

And so $z \cdot \bar{z} = N(z)$. ∎

$\boxed{\#4}$ $N(zw) = \overline{zw}\, zw = \bar{z} z \cdot \bar{w} w = N(z) N(w)$.

I've used that $\overline{zw} = \bar{z}\,\bar{w}$, which we'll check now. Write $z = a+bi$, $w = u+vi$.

Then $\overline{zw} = \overline{(a+bi)(u+vi)} = \overline{au + avi + bui - bv}$

$= au - bv - avi - bui$

$= (au-bv) - (av+bu)i$.

OTOH: $\bar{z}\,\bar{w} = \overline{(a+bi)} \cdot \overline{(u+vi)} = (a-bi)(u-vi) =$

$= (au-bv) - (av+bu)$.

So indeed, $\overline{zw} = \bar{z}\,\bar{w}$. ∎ (And the one line at top gives the proof).

$\boxed{36.2}$

(a) $\alpha = 11+17i, \quad \beta = 5+3i$

Then $$\frac{11+17i}{5+3i}\,\frac{(5-3i)}{(5-3i)} = \frac{106+52i}{34} = \frac{106}{34} + \frac{52}{34}i.$$

The closest integer to $\frac{106}{34}$ is 3, + to $\frac{52}{34}$ is 2.

So $q = 3+2i$, and $r = (11+17i) - (3+2i)(5+3i) = 2-2i$.

Note that $N(r) = N(2-2i) = 8 < 34 = N(\beta)$. //

(b) $\alpha = 12-23i, \quad \beta = 7-5i$

$$\frac{12-23i}{7-5i}\,\frac{(7+5i)}{(7+5i)} = \frac{199-101i}{74} = \frac{199}{74} - \frac{101}{74}i \approx 3-i.$$

So $q = 3-i$, $\quad r = (12-23i)-(7-5i)(3-i) = -4-i$.

And $N(r) = 17 < 74 = N(\beta)$.

(c) $\alpha = 21-20i, \quad \beta = 3-7i$

$$\frac{21-20i}{3-7i}\,\frac{(3+7i)}{(3+7i)} = \frac{203+87i}{58} = \frac{203}{58} + \frac{87}{58}i$$

$$\approx 3+i$$

(But notice, really its $3.5+1.5i$, really ambiguous! Lots of choices).

So $q = 3+i$, $r = (21-20i)-(3-7i)(3+i) = 5-2i$,

$N(r) = 29 < 58 = N(\beta)$.

(the last one has 4 different choices for q, r).

36.4 (a) We showed this directly in class.

(b) from Bezout's Theorem, which we proved in class by using reverse substitution in the Euclidean Alg, we know that $\gcd(z,w) = zx + wy$ has solutions for $x, y \in \mathbb{Z}[i]$, and so the set $\{zx + wy : x, y \in \mathbb{Z}[i]\}$ contains all the gcds of z, w.

(c) Let γ be a $\gcd(z,w)$. Then as $\gamma \mid z$, $\gamma \mid w$, we have $\gamma \mid zx + wy$ for all $x, y \in \mathbb{Z}[i]$. So every number in $\{zx+wy\}$ is a multiple of γ.
Conversely, given any multiple $t\gamma$ of γ, we see there is a solution to $zx+wy = t\gamma$ (by multiplying a solution to $zx+wy=\gamma$ by t).
So any multiple of γ is in the set.

Thus $\{zx+wy : x, y \in \mathbb{Z}[i]\}$ is exactly $\{t\gamma : t \in \mathbb{Z}[i]\}$.

Note: We proved the analogous statment for $\mathbb{Z}$, but with different wording. Do you remember?

Now, in $\mathbb{Z}[\sqrt{-5}]$.

(#7) $3+2\sqrt{-5} \mid 85-11\sqrt{-5}$, which we figure out naively.

$$(3+2\sqrt{-5})(a+b\sqrt{-5}) = 85-11\sqrt{-5}$$

$$3a-5\cdot 2b + (3b+2a)\sqrt{-5} = 85-11\sqrt{-5}.$$

$$\Longrightarrow \begin{cases} 3a-10b = 85 \\ 3b+2a = -11 \end{cases}$$

$$\Longrightarrow a=5, \quad b=-7.$$

So in fact, $85-11\sqrt{-5} = (3+2\sqrt{-5})(5-7\sqrt{-5})$,

so clearly $3+2\sqrt{-5} \mid 85-11\sqrt{-5}$. ∎

(#8) We verify directly. $a = u+v\sqrt{-5}$, $b = s+t\sqrt{-5}$.

Then $N(a)N(b) = (u^2+5v^2)(s^2+5t^2) = u^2s^2+5u^2t^2+5v^2s^2+25v^2t^2$.

$a\cdot b = us-5vt+(ut+vs)\sqrt{-5}$.

$$N(ab) = (us-5vt)^2+5(ut+vs)^2$$

$$= u^2s^2 - \cancel{10uvts} + 25v^2t^2 + 5u^2t^2 + \cancel{10utvs} + 5v^2s^2$$

$$= u^2s^2+5u^2t^2+5v^2s^2+25v^2t^2,$$ which is the same as .

So $N(a)N(b) = N(ab)$. ∎

#9 $6 = 2\cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$.

$N(2) = 4$, $N(3) = 9$, $N(1\pm\sqrt{-5}) = 6$.

Recall that $a \mid b \Rightarrow N(a) \mid N(b)$.

As neither 4 or 9 divide 6 (or the other way around), none of the 4 numbers

$$2, 3, \quad 1+\sqrt{-5}, \quad 1-\sqrt{-5},$$

divide the others. ∎

#10 No! In fact, this is an example of a number with 2 distinct factorizations into irreducibles, $6 = 2\cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$. ∎

We mentioned in class that each of these are irreducibles, but let's see one way to prove that now.

If 2 factored, then $2 = ab$ with $N(a), N(b) > 1$.

As $N(2) = 4$, we must have $N(a) = N(b) = 2$.

So we try to solve $x^2 + 5y^2 = 2$ ~~(representing $a = x+y\sqrt{-5}$)~~ (representing $a = x+y\sqrt{-5}$).

~~Clearly~~ Clearly, $y = 0$ (otherwise, the left-hand side is too large), leading to $x^2 = 2$. But this has no solution! So 2 is irreducible. One can repeat this style argument for the other numbers too. //