On Some Variants of the Gauss Circle Problem

by
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The Gauss Circle Problem concerns finding asymptotics for the number of lattice point lying inside a circle in terms of the radius of the circle. The heuristic that the number of points is very nearly the area of the circle is surprisingly accurate. This seemingly simple problem has prompted new ideas in many areas of number theory and mathematics, and it is now recognized as one instance of a general phenomenon. In this work, we describe two variants of the Gauss Circle problem that exhibit similar characteristics.

The first variant concerns sums of Fourier coefficients of GL(2) cusp forms. These sums behave very similarly to the error term in the Gauss Circle problem. Normalized correctly, it is conjectured that the two satisfy essentially the same asymptotics.

We introduce new Dirichlet series with coefficients that are squares of partial sums of Fourier coefficients of cusp forms. We study the meromorphic properties of these Dirichlet series and use these series to give new perspectives on the mean square of the size of sums of these Fourier coefficients. These results are compatible with current conjectures.

The second variant concerns the number of lattice points of bounded size on one-sheeted hyperboloids. This problem is very similar to counting the number of lattice points within a spheres of various dimensions, except with the additional constraint of lying on a hyperboloid. It turns out that this problem is equivalent to estimating sums of the shape $\sum r_d(n^2 + h)$, where $r_d(m)$ is the number of representations of m as a sum of d squares. We prove improved smoothed and sharp estimates of the second moment of these sums, yielding improved estimates of the number of lattice points.

In both variants, the problems are related to modular forms and, in particular, to shifted convolution sums associated to these modular forms. We introduce new techniques and new Dirichlet series which are quite general. At the end of this work, we describe further extensions and questions for further investigation.

This dissertation by David Lowry-Duda is accepted in its present form
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Publications

- Alina Bucur, Edgar Costa, Chantal David, João Guerreiro, David Lowry-Duda, *Traces, High powers and One level density for families of curves over finite fields*, To appear in the Mathematical Proceedings of the Cambridge Philosophical Society.
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  Preprint: http://arxiv.org/abs/1512.01299.

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  Preprint: http://arxiv.org/abs/1512.05502.

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  DOI: 10.4169/college.math.j.46.1.56.

Presentations

- *On Problems Related to the Gauss Circle Problem*, Dartmouth Number Theory Seminar, Dartmouth, NH, March 2017.

- *Counting Points on One-Sheeted Hyperboloids*, MSRI Number Theory Seminar, Berkeley, CA, February 2017.

- *Old and New Results on the Gauss Circle Problem*, Louisiana Tech, LA, January 2017.

- Lowry-Duda, D. On Iterated Partial Sums of Coefficients of Cusp Forms and Related Topics. In: 2016 Maine-Quebec Conference, Université Laval, Québec, October 8–9, 2016.

- *Functions where midpoints are mean value abscissas, and other questions concerning the (not so boring) mean value theorem of calculus*, Graduate Student Seminar, Brown University, RI, April 2016.
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Dedication

To a string of educators with big hearts, nurturing a childish curiosity in all things:

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Much of the work in this thesis was either directly or indirectly inspired by collaboration with my mathematical family. Chapter 3 was born out of a question asked by Jeff Hoffstein at his 61st birthday conference concerning the sign changes of sums of coefficients of cuspforms, which I thought about immediately afterwards with Tom Hulse, Chan Ieong Kuan, and Alex Walker. This eventually led to both the ideas of Chapter 3 and the papers briefly described in Chapter 4. The main question of Chapter 5 was brought to my attention by Chan Ieong Kuan and Mehmet Kıral several years ago. During our first (somewhat misguided) attempts, I learned a lot about mathematics and collaboration, and I owe them much.

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Contents

Chapter One

Introduction

> Charles Darwin had a theory that once in a while one should perform a damn-fool experiment. It almost always fails, but when it does come off is terrific.
>
> Darwin played the trombone to his tulips. The result of this particular experiment was negative.
>
> Littlewood, *A Mathematician's Miscellany*

In this thesis, we study and introduce new methods to study problems that are closely related to the Gauss Circle problem. In this introductory chapter, we motivate and explain the Gauss Circle problem and how it relates to the other problems described in later chapters. In Section 1.1, we give a historical overview of the classical Gauss Circle and Dirichlet Hyperbola problems.

1.1 Gauss Circle Problem

The Gauss Circle problem is the following seemingly innocent question:

> How many integer lattice points lie inside or on the circle of radius $\sqrt{R}$ centered at the origin? That is, how many points $(x, y) \in \mathbb{Z}^2$ satisfy $x^2 + y^2 \leq R$?

We will use $S_2(R)$ to denote the number of lattice points inside or on the circle of radius $\sqrt{R}$ centered at the origin. Intuitively, it is clear that $S_2(R)$ should be approximately the area of the circle, $\operatorname{Vol} B(\sqrt{R}) = \pi R$. This can be made rigorous by thinking of each lattice point as being the center of a 1×1 square in the plane and counting those squares fully contained within the circle and those squares lying on the boundary of the circle. Using this line of thought, Gauss proved (in 1798) that

$$|S_2(R) - \operatorname{Vol} B(\sqrt{R})| \ll \sqrt{R},$$

so that the *lattice point discrepancy* between the number of lattice points within the circle and the area of the circle is bounded by some constant times the circumference.

It may appear intuitive that this is the best one could hope for.[†] No further progress was made towards Gauss Circle problem until 1906, when Sierpiński [Sie06]

[†]Indeed, I am not currently aware of an intuitive heuristic that explains *why* the actual error term is so much better.

showed that

$$|S_2(R) - \operatorname{Vol} B(\sqrt{R})| \ll R^{\frac{1}{3}},$$

a significant improvement over the bound from Gauss. This is already remarkable — somehow lattice points in the plane are distributed in the plane in a way that a growing circle includes and omits points in a way that partially offsets each other.

It is natural to ask: What is the correct exponent of growth on the error term? Hardy, Littlewood, and Cramér [Cra22] proved that, *on average*, the correct exponent is $\frac{1}{4}$ when they proved

$$\int_0^X |S_2(r) - \operatorname{Vol} B(\sqrt{r})|^2 dr = cX^{\frac{3}{2}} + O(X^{\frac{5}{4}+\epsilon}) \tag{1.1}$$

for some constant c. At approximately the same time, Hardy [Har17] showed that

$$|S_2(R) - \operatorname{Vol} B(\sqrt{R})| = \Omega(R^{\frac{1}{4}}),$$

so that $\frac{1}{4}$ is both a lower bound and the average bound.

At the same time (and often, even within the same works, such as in Hardy's work on the average and minimum values of the error in [Har17]), mathematicians were studying Dirichlet's Divisor problem. Let $d(n)$ denote the number of positive divisors of n. Then Dirichlet's Divisor problem is to determine the average size of $d(n)$ on integers n up to R. Noting that

$$\sum_{n \le R} d(n) = \sum_{n \le R} \sum_{d|n} 1 = \sum_{d \le R} \left\lfloor \frac{R}{d} \right\rfloor,$$

we see that this is equivalent to counting the number of positive integer lattice points under the hyperbola $xy = R$. For this reason, some refer to this as Dirichlet's Hyperbola problem.

As noted already by Hardy [Har17], it's known that

$$\sum_{n \le R} d(n) = cR \log R + c'R + O(R^{\frac{1}{2}}),$$

and that

$$\int_1^X \left| \sum_{n \le r} d(n) - cr \log r - c'r \right|^2 dr = c''X^{\frac{3}{2}} + O(X^{\frac{5}{4}+\epsilon}),$$

analogous to the Gauss Circle problem. Attempts to further understand the error terms in the Gauss Circle and Dirichlet Hyperbola problems have indicated that there is a strong connection between the two, and often an improvement to one of the problems yields an improvement to the other.

It is interesting to note that much of the early work of Hardy and Littlewood on the Gauss Circle and Dirichlet Hyperbola problems occurred from 1914 to 1919, which are the years when Ramanujan studied and worked with them at Cambridge. Of particular import is a specific identity inspired by Ramanujan (as noted by Hardy in [Har59]) that is now sometimes called "Hardy's Identity"[‡], which states that

$$S_2(R) - \text{Vol}\, B(\sqrt{R}) = \sqrt{R} \sum_{n \geq 1} \frac{r_2(n)}{n^{1/2}} J_1(2\pi\sqrt{nR}), \tag{1.2}$$

in which $r_2(n)$ is the number of ways of writing n as a sum of 2 squares and J_ν is the ordinary Bessel function

$$J_\nu(z) := \sum_{n \geq 0} \frac{(-1)^n}{\Gamma(n+1)\Gamma(\nu+n+1)} (z/2)^{\nu+2n}.$$

Ivić has noted in [IKKN04] and [Ivi96] that almost all significant progress towards both the Gauss Circle and Dirichlet Hyperbola problems have come from identities and approaches similar to (1.2), though sometimes obscured through technical details.

1.1.1 An Early Connection to Modular Forms

One of the topics that Ramanujan devoted himself to was what we now call the "Ramanujan tau function," which can be defined by matching coefficients in

$$\sum_{n \geq 1} \tau(n) q^n = q \prod_{n \geq 1} (1 - q^n)^{24}.$$

Ramanujan noticed or conjectured that this function satisfies many nice properties, such as being multiplicative. The individual $\tau(n)$ satisfy the bound

$$\tau(n) \ll n^{\frac{11}{2}+\epsilon},$$

and numerical experimentation might lead one to conjecture that

$$\sum_{n \leq R} \tau(n) \ll R^{\frac{11}{2}+\frac{1}{4}+\epsilon}.$$

Just like the Gauss Circle and Dirichlet Hyperbola problem, it appears that summing over R many terms contributes only $\frac{1}{4}$ to the exponent in the size. Further, Chandrasekharan and Narasimhan [CN62, CN64] showed a result analogous to the average estimate of Hardy and Littlewood, proving that

$$\int_0^X \Big| \sum_{n \leq r} \tau(n) \Big|^2 dr = cX^{11+\frac{3}{2}} + O(X^{12+\epsilon}).$$

[‡]This is another instance of Stigler's Law of Eponymy.

This also indicates that the average additional exponent is $\frac{1}{4}$.

We now recognize that this is another analogy to the Gauss Circle problem, except that in this case there is no main term. This analogy readily generalizes.

The Ramanujan τ function appears as the coefficients of the unique weight 12 holomorphic cusp form on $\mathrm{SL}(2,\mathbb{Z})$, usually written

$$\Delta(z) = \sum_{n\geq 1} \tau(n) e(nz),$$

where here and throughout this thesis, $e(nz) = e^{2\pi i n z}$. Generally, we can consider a holomorphic cusp form f on a congruence subgroup $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ of weight k, with Fourier expansion

$$f(z) = \sum_{n\geq 1} a(n) e(nz).$$

Let $S_f(R)$ denote the partial sum of the first R Fourier coefficients,

$$S_f(R) := \sum_{n\leq R} a(n).$$

Then Chandrasekharan and Narasimhan also show that

$$\int_0^X |S_f(r)|^2 dr = cX^{k+\frac{3}{2}} + O(X^{k+\epsilon}),$$

indicating that *on average*, the partial sums satisfy

$$S_f(R) \ll R^{\frac{k-1}{2}+\frac{1}{4}+\epsilon}.$$

In other words, partial sums of coefficients of cusp forms appear to satisfy a Gauss Circle problem type growth bound.

Further, the Gauss Circle problem can be restated as a problem estimating

$$S_2(R) = \sum_{n\leq R} r_2(n),$$

where $r_2(n)$ denotes the number of representation of n as a sum of two squares as above. The coefficients $r_2(n)$ appear as the coefficients of a the modular form $\theta^2(z)$, so the analogy between S_f and S_2 is very strong. However θ^2 is not cuspidal, so there are some differences.

In Chapter 3, we consider this "Cusp Form Analogy" and study $S_f(n)$. We introduce new techniques that are fundamentally different than most techniques

employed in the past. Of particular interest is the introduction of Dirichlet series of the form

$$\sum_{n\geq 1} \frac{S_f(n)^2}{n^s},$$

including their meromorphic continuation and analysis. In Chapter 4, we give an overview of the completed and currently planned applications of the analysis and techniques for studying $S_f(n)$ in Chapter 3. This includes an overview of several recent papers of the author and his collaborators where these techniques have been used successfully.

1.1.2 Further Generalizations of the Gauss Circle Problem

A very natural generalization of the Gauss Circle problem is to higher dimensions. We will call the following the Gauss d-Sphere problem:

> How many integer lattice points lie inside or on the d-dimensional sphere of radius $\sqrt{R}$ centered at the origin? That is, how many points $\boldsymbol{x} \in \mathbb{Z}^d$ satisfy $\|\boldsymbol{x}\|^2 \leq R$?

We use $S_d(R)$ to denote the number of lattice points inside or on the d-sphere of radius $\sqrt{R}$, as occurs in the Gauss d-Sphere problem. As with the Gauss Circle problem, it is intuitively clear that $S_d(R) \approx \operatorname{Vol} B_d(\sqrt{R})$, where we use $B_d(\sqrt{R})$ to denote a d-dimensional ball of radius $\sqrt{R}$ centered at the origin. Just as with the Gauss Circle problem, the true goal of the Gauss d-Sphere problem is to understand the lattice point discrepancy $S_d(R) - \operatorname{Vol} B_d(\sqrt{R})$.

The Gauss d-Sphere problem is most mysterious for low dimensions, but perhaps the dimension 3 Gauss Sphere problem is the most mysterious. For an excellent survey on the status of this problem, see [IKKN04].

The Gauss d-Sphere problem is not considered in detail in this thesis, but in Chapter 4 a new approach on aspects of the Gauss d-Sphere problem is outlined which builds on the techniques of Chapter 3. This is a project under current investigation by the author and his collaborators.

Closely related to the Gauss d-Sphere problem is the problem of determining the number of lattice points that lie within $B_d(\sqrt{R})$ **and** which lie on a one-sheeted hyperboloid

$$\mathcal{H}_{d,h} := X_1^2 + \cdots + X_{d-1}^2 = X_d^2 + h$$

for some $h \in \mathbb{Z}_{\geq 1}$. We use $N_{d,h}(R)$ to denote the number of lattice points within $B_d(\sqrt{R})$ and on $\mathcal{H}_{d,h}$. This is a problem within the larger class of constrained lattice counting problems.

The dimension 3 one-sheeted hyperboloid problem is the first nontrivial dimension, and just as with the standard Gauss d-Sphere problem, the dimension 3 hyperboloid is the most enigmatic. Very little is currently known. A significant reason for the mystery comes from the fact that this is too small of a dimension to apply the Circle Method of Hardy and Littlewood, and there doesn't seem to be an immediate analogue of Hardy's Identify (1.2). In fact, it's not quite clear what the correct separation between the main term and error term should be. The best known result is the recent result of Oh and Shah [OS14], which uses ergodic methods to prove that

$$N_{3,h}(R) = cR^{\frac{1}{2}} \log R + O\big(R^{\frac{1}{2}} (\log R)^{\frac{3}{4}}\big) \tag{1.3}$$

when h is a positive square.

From the perspective of modular forms, the underlying modular object is the modular form $\theta^{d-1}(z)\overline{\theta(z)}$. But in contrast to the variants of the Gauss Circle problem discussed above, $N_{d,h}(R)$ is encoded within the hth Fourier coefficient of $\theta^{d-1}\overline{\theta}$, which affects the shape of the analysis significantly.

In Chapter 5, we consider the problem of estimating the number $N_{d,h}(R)$ of points on one-sheeted hyperboloids. Along the way, we study the Dirichlet series

$$\sum_{n \geq 1} \frac{r_{d-1}(n^2 + h)}{(2n^2 + h)^s}$$

and its meromorphic properties, and use it to get improved estimates for $N_{d,h}(R)$.

1.2 Outline and Statements of Main Results

Chapter 2.

Chapter 2 consists of background information used in later sections. In particular, a variety of properties concerning Eisenstein series are discussed and referenced from the literature. The second half of Chapter 2 concerns a complete description of three Mellin integral transforms. It is shown that two smooth integral transforms can be used in tandem to establish a sharp cutoff result, which is employed in Chapter 5 and in the applications described in Chapter 4.

Chapters 3 and 4.

Chapter 3 is closely related to the journal article [HKLDW17a], which was published in 2017. In this Chapter, we introduce and study the meromorphic properties of the Dirichlet series

$$\sum_{n\geq 1} \frac{S_f(n)}{n^s}, \quad \sum_{n\geq 1} \frac{S_f(n)\overline{S_g(n)}}{n^s}, \quad \text{and} \quad \sum_{n\geq 1} \frac{S_f(n)S_g(n)}{n^s}, \tag{2.1}$$

where f and g are weight k cuspforms on a congruence subgroup $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ and where $S_f(n)$ and $S_g(n)$ denote the partial sums of the first n Fourier coefficients of f and g, respectively.

The first major result is in Theorem 3.4.8 and Corollary 3.4.9, which show that the series in (2.1) have meromorphic continuation to the plane with understandable analytic properties.

As a first application of this meromorphic continuation, we prove the major analytic result of the chapter in Theorem 3.5.3, which states that

$$\sum_{n\geq 1} \frac{S_f(n)\overline{S_g(n)}}{n^{k-1}} e^{-n/X} = CX^{\frac{3}{2}} + O(X^{\frac{1}{2}+\epsilon}),$$

and related results, for an explicit constant C. In terms of the analogy to the Gauss Circle problem, this is a smoothed second moment and is comparable in nature to the mean-square moment bound of Hardy and Littlewood (1.1). As the Dirichlet series in (2.1) are new, this smoothed result is a new type of result.

Chapter 4 is a description of further work using the Dirichlet series (2.1) and its meromorphic continuation. This includes the work in the papers [HKLDW17c] and [HKLDW16]. Further applications are described, as well as preliminary results from the author and his collaborators on this topic.

Chapters 5 and 6

Chapter 5 focuses on studying the One-Sheeted Hyperboloid problem. The Dirichlet series

$$\sum_{n\in\mathbb{Z}} \frac{r_{d-1}(n^2+h)}{(2n^2+h)^s}$$

is studied, and the first major result of the Chapter is in Theorem 5.3.13, which states that this Dirichlet series has meromorphic continuation to the plane with understandable analytic behavior.

Using this meromorphic continuation, three applications are given. Firstly, there is Theorem 5.5.1, which gives a smoothed analogue of the number of lattice points $N_{d,h}(R)$ on the one-sheeted hyperboloid $\mathcal{H}_{d,h}$ and within $B_d(\sqrt{R})$, including many second order main terms. This Theorem can be interpreted as a long-interval smoothed average, with many secondary main terms.

Secondly, there is Theorem 5.5.3, which proves a short-interval sharp average result. But the most important result of the chapter is Theorem 5.5.4, which improves the state of the art estimate of Oh and Shah in (1.3) by showing that

$$N_{3,h}(R) = C'R^{\frac{1}{2}} \log R + CR^{\frac{1}{2}} + O(R^{\frac{1}{2}-\frac{1}{44}+\epsilon})$$

when h is a positive square. When h is not a square, the $R^{\frac{1}{2}} \log R$ term does not appear. In fact, the theorem is very general and gives results for any dimension $d \geq 3$.

In Chapter 6, we describe an application of the methods and techniques of Chapter 5 to asymptotics of sums of the form

$$\sum_{n \leq X} d(n^2 + h),$$

where $d(m)$ denotes the number of positive divisors of m.

Chapter Two

Background

In this chapter, we review and describe some topics that will be used heavily in later chapters of this thesis. This chapter should be used as a reference. The topics covered are classical and well-understood, but included for the sake of presenting a complete idea.

2.1 Notation Reference

We use the basic notation of Landau, so that

$$F(x) \ll G(x)$$

means that there are constants C and X such that for all $x > X$, we have that

$$|F(x)| \leq CG(x).$$

We use $F(x) \ll G(x)$ and $F(x) = O(G(x))$ interchangeably. We also use

$$F(x) = o(G(x))$$

to mean that for any $\epsilon > 0$, there exists an X such that for all $x > X$, we have that $F(x) \leq \epsilon G(x)$. On the other hand,

$$F(x) = \Omega(G(x))$$

means that F and G do not satisfy $F(x) = o(G(x))$. Stated differently, $F(x)$ is as least as large as $G(x)$ (up to a constant) infinitely often.

We will use $r_d(n)$ to denote the number of representations of n as a sum of d squares, i.e.

$$r_d(n) := \#\{\boldsymbol{x} \in \mathbb{Z}^d : x_1^2 + \cdots + x_d^2 = n\}.$$

We will denote partial sums of $r_d(n)$ by S_d, i.e.

$$S_d(X) := \sum_{n \leq X} r_d(n).$$

We use $B(R)$ to denote the disk of radius R centered at the origin. It is traditional to talk about the *Gauss Circle problem* instead of the *Gauss Disk problem*, and so we will frequently refer to $B(R)$ as a circle. We use $B_d(R)$ to denote the dimension d ball of radius R centered at the origin, and for similar conventional reasons we will frequently refer to $B_d(R)$ as a sphere. Throughout this work, we will never distinguish between points on the surface of a sphere and those points contained within the sphere, so we use these terms interchangeably.

With respect to classical modular forms, we typically use the conventions and notations of [Bum98, Gol06].

Typically, f will be a weight k holomorphic modular form on a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$. By this, we mean the following. A matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{R})$ acts on z in the upper half plane $\mathcal{H}$ by

$$\gamma z = \frac{az+b}{cz+d}.$$

A holomorphic modular form of (integral) weight k on a congruence group $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ is a holomorphic function $f : \mathcal{H} \longrightarrow \mathbb{C}$, which satisfies

$$f(\gamma z) = (cz+d)^k f(z) \qquad \text{for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma, \tag{1.1}$$

and which is holomorphic at ∞. The latter condition requires some additional clarification, and an excellent overview of this concept is in the first chapter of Diamond's introductory text [DS05]. In the context of this thesis, this latter condition guarantees that f has a Fourier expansion of the form

$$f(z) = \sum_{n \geq 0} a(n) e(nz),$$

where $e(nz) = e^{2\pi i n z}$. If in addition $a(0) = 0$, so that the Fourier expansion does not have a constant term, then we call f a holomorphic cusp form.

Note that some authors adopt the convention of writing the Fourier coefficients as $a(n) = A(n) n^{\frac{k-1}{2}}$, which is normalized so that $A(n) \approx 1$ on average. We do not use normalized coefficients in this thesis.

When it is necessary to use a second holomorphic modular form, we will denote it by g and its Fourier expansion by

$$g(z) = \sum_{n \geq 1} b(n) e(nz).$$

Partial sums of these coefficients are denoted by

$$S_f(X) := \sum_{n \leq X} a(n), \qquad S_g(X) := \sum_{n \leq X} b(n).$$

In Chapter 5, we will use half-integral weight modular forms extensively. Let $j(\gamma, z)$ be defined as

$$j(\gamma, z) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (cz+d)^{\frac{1}{2}}, \qquad \gamma \in \Gamma_0(4),$$

where $\varepsilon_d = 1$ if $d \equiv 1 \bmod 4$ and $\varepsilon_d = i$ if $d \equiv 3 \bmod 4$. This is the same transformation law that is satisfied by the Jacobi theta function $\theta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z}$, which is the prototypical half-integral weight modular form. Then a half-integral weight modular form of weight k on $\Gamma_0(4)$ is defined in the same was as a full-integral weight modular form, except that it satisfies the transformation law

$$f(\gamma z) = j(\gamma, z)^{2k} f(z) \qquad \text{for } \gamma \in \Gamma_0(4)$$

instead of (1.1). Note that we will use k to denote a full integer or a half integer in reference to "modular forms of weight k." This is a different convention than some authors, who use $k/2$ in discussion of half-integral weight forms.

Given two modular forms f and g defined on a congruence subgroup $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$, we will let $\langle f, g\rangle$ denote the Petersson inner product

$$\langle f, g\rangle := \iint_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)} d\mu(z).$$

In this expression, $\mathcal{H}$ denotes the complex upper half plane and $d\mu(z)$ denotes the Haar measure, which in this case is given by

$$d\mu(z) = \frac{dx\, dy}{y^2}.$$

We will use $\left(\frac{a}{n}\right)$ to denote the Jacobi symbol. Note that this sometimes may look very similar to a fraction. Generally, the Jacobi symbol involves only arithmetic data, while fractions will have complex valued arguments.

We also roughly follow some conventions concerning variable names.

The primary indices of summation will be m and n. If there is a shifted summation, we will usually use h to denote the shift. We will sometimes use ℓ to denote a distinguished index of summation (or more generally a distinguished variable in local discussion). The one major exception to this convention is the index j, which we reserve for spectral summations (and in particular the discrete spectrum) or as an index for residues and residual terms appearing from spectral analysis.

Complex variables will be denoted by z, s, w, and u. Almost always $z = x + iy$ will be in the upper half plane $\mathcal{H}$, and denotes the complex variable of the underlying modular form. The other variables, s, w, and u denote generic complex variables and usually appear as the variables of various Dirichlet series. We often follow the odd but classical convention of mixing Roman and Greek characters, and write $s = \sigma + it$. Note that we sometimes use t to denote a generic real variable in integrals.

If $L(s)$ is an L-function, then $L^{(q)}(s)$ denotes that L -function, but with the Euler factors corresponding to primes p dividing q removed.

We use $\mathfrak{a}$ to denote a cusp of a modular curve. In this thesis, this almost always means that $\mathfrak{a}$ is one of the three cusps of $\Gamma_0(4)\backslash\mathcal{H}$.

2.2 On Full and Half-Integral Weight Eisenstein Series

The primary goal of this section is to describe some characteristics of the weight k Eisenstein series for $\Gamma_0(4)$, both for full-integral weight and half-integral weight k. These details are often considered standard exercises in the literature, and are usually tedious to compute. We will emphasize the important properties of the Eisenstein series necessary for the argument in Chapter 5 — some with complete proofs, and some with descriptions of the method of proof.

Selberg defined the weight 0 classical real analytic Eisenstein series $E(z,w)$ on $\mathrm{SL}(2,\mathbb{Z})$ by

$$E(z,w) = \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})} \mathrm{Im}(\gamma z)^s.$$

The weight 0 Eisenstein series $E(z,w)$ is very classical and very well-understood, and both [Gol06, Chapter 3] and [Iwa97, Chapter 13] provide an excellent description of its properties.

Weight k Eisenstein series are also very classical, but they appear less often and with much less exposition in the literature. Half-integral weight Eisenstein series occupy an even smaller role in the literature, although in recent years the study of metaplectic forms and metaplectic Eisenstein series has grown in popularity.

We will use $E^k_{\mathfrak{a}}(z,w)$ to denote the weight k Eisenstein series associated to the cusp $\mathfrak{a}$. In this expression, k can be either a half-integer or a full-integer. In this thesis, we use the half-integral weight Eisenstein series on the congruence subgroup $\Gamma_0(4)$. The quotient $\Gamma_0(4)\backslash\mathcal{H}$ has three cusps, at $\infty, 0$, and $\frac{1}{2}$.

We shall describe the three Eisenstein series $E^k_\infty(z,w), E^k_0(z,w)$, and $E^k_{\frac{1}{2}}(z,w)$, both for full-integral weight and half-integral weight k. These Eisenstein series are defined as

$$E^k_\infty(z,w) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} \mathrm{Im}(\gamma z)^w J(\gamma,z)^{-2k} = \sum_{\substack{\pm(c,d)\in\mathbb{Z}^2\\(c,d)\neq(0,0)\\\gcd(4c,d)=1}} \frac{y^w \varepsilon_d^{2k}\left(\frac{4c}{d}\right)^{2k}}{|4cz+d|^{2w-k}(4cz+d)^k}$$

$$E_0^k(z,w) = \left(\frac{z}{|z|}\right)^{-k} E_\infty^k(\tfrac{-1}{4z}, w) = \sum_{\substack{\pm(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0) \\ \gcd(4c,d)=1}} \frac{(y/4)^w \varepsilon_d^{2k} \left(\frac{4c}{d}\right)^{2k}}{|-c+dz|^{2w-k}(-c+dz)^k}$$

$$E_{\frac{1}{2}}(z,w) = \left(\frac{2z+1}{|2z+1|}\right)^{-k} E_\infty^k(\tfrac{z}{2z+1}, w) = \sum_{\substack{\pm(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0) \\ \gcd(2c,d)=1}} \frac{y^w \varepsilon_d^{2k} \left(\frac{2c}{d}\right)^{2k}}{|2cz+d|^{2w-k}(2cz+d)^k}.$$

In these expressions, ε_d denotes the sign of the Gauss sum associated to the primitive real quadratic Dirichlet character χ_d, and is given by

$$\varepsilon_d = \begin{cases} 1 & d \equiv 1 \bmod 4, \\ i & d \equiv 3 \bmod 4. \end{cases}$$

We use $J(\gamma, z)$ to denote the normalized half-integral weight cocycle $J(\gamma, z) = j(\gamma, z)/|j(\gamma, z)|$, where $j(\gamma, z)$ is the standard θ cocycle

$$j(\gamma, z) := \theta(\gamma z)/\theta(z) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (cz+d)^{\frac{1}{2}}.$$

Here, $\theta(z)$ denotes the Jacobi theta function, which is a modular form of weight $\frac{1}{2}$ on $\Gamma_0(4)$. Thus this cocycle law holds for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$.

Notice that when k is a full integer, the weighting factor simplifies to

$$\varepsilon_d^{2k} \left(\frac{4c}{d}\right)^{2k} = \left(\frac{-1}{d}\right)^k,$$

and when k is a half-integer, the weighting factor simplifies slightly to

$$\varepsilon_d^{2k} \left(\frac{4c}{d}\right)^{2k} = \varepsilon_d^{2k} \left(\frac{4c}{d}\right),$$

(where in both cases the 4 is replaced by 2 for $E_{\frac{1}{2}}^k$). This is a restatement of the fact that when k is a full integer, the transformation law is full-integral with character $\left(\frac{-1}{\cdot}\right)^k$, while for half-integers the transformation law is a power of a standard normalized theta cocycle.

When k is a half-integer, these three Eisenstein series are essentially the same three Eisenstein series that appear in [GH85], only with some notational differences. Firstly, in [GH85], half-integer weights are denoted by $\frac{k}{2}$, while in this work we denote all weights (both integral and half-integral) by k. Secondly, we shift the spectral argument, replacing w with $w - \frac{k}{2}$ as compared to [GH85]. This has the effect of using the normalized cocycle $J(\gamma, z)$ instead of $j(\gamma, z)$, and also normalizes the arguments of the L-functions that appear in the coefficients of the Eisenstein series.

Each Eisenstein series has a Fourier-Whittaker expansion of the form

$$E_{\mathfrak{a}}^k(z,w) = y^w + \rho_{\mathfrak{a}}^k(0,w)y^{1-w} + \sum_{h\neq 0} \rho_{\mathfrak{a}}^k(h,w) W_{\frac{|h|}{h}\frac{k}{2}, w-\frac{1}{2}}(4\pi|h|y)e^{2\pi i h x}$$

for some coefficients $\rho_{\mathfrak{a}}^k(h,w)$, and where $W_{\alpha,\nu}(y)$ is the GL(2) Whittaker function. This is given by [GH11, 3.6.3]

$$W_{\alpha,\nu}(y) = \frac{y^{\nu+\frac{1}{2}}e^{-\frac{y}{2}}}{\Gamma(\nu-\alpha+\frac{1}{2})}\int_0^\infty e^{-yt}t^{\nu-\alpha-\frac{1}{2}}(1+t)^{\nu+\alpha-\frac{1}{2}}dt,$$

valid for $\mathrm{Re}(\nu-\alpha) > \frac{-1}{2}$ and $|\arg(y)| < \pi$.

When $\alpha = 0$, the Whittaker function is just the K-Bessel function

$$W_{0,\nu}(y) = \big(\frac{y}{\pi}\big)^{\frac{1}{2}}K_\nu\big(\frac{y}{2}\big), \quad K_\nu(y) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}y(u+u^{-1})}u^\nu\frac{du}{u}.$$

Thus in the weight 0 case, the Whittaker functions simplify to K-Bessel functions (as noted and employed in [Gol06] for instance). This additional difficulty in weight k is perhaps one reason why most expository accounts stop at the weight 0 case.

According to the general theory of Selberg (and described in [Iwa97, Theorem 13.2]), the potential poles of $E_{\mathfrak{a}}^k(z,w)$ for $w > \frac{1}{2}$ can be recognized from the poles of the constant coefficient $\rho_{\mathfrak{a}}^k(0,w)$.

Although the individual expansions vary, they are usually of a very similar form. For full integral weight, the coefficients have the shape

$$\begin{aligned}\rho_{\mathfrak{a}}^k(0,w) &= (*)\frac{L_{\mathfrak{a}}^k(2w-1)}{L_{\mathfrak{a}}^k(2w)}\frac{\Gamma(2w-1)}{\Gamma(w+\frac{k}{2})\Gamma(w-\frac{k}{2})}\\ \rho_{\mathfrak{a}}^k(h,w) &= (*)\frac{|h|^{w-1}}{L_{\mathfrak{a}}^k(2w)}\frac{1}{\Gamma(w+\frac{|h|}{h}\frac{k}{2})}D_{\mathfrak{a}}^k(h,w)\end{aligned}$$

where $(*)$ is a (possibly zero) constant times a collection of powers of 2 and π, $L_{\mathfrak{a}}^k(s)$ is a GL(1) L-function (maybe missing some Euler factors), and $D_{\mathfrak{a}}^k$ is a finite Dirichlet sum. For $\mathrm{Re}\, w > \frac{1}{2}$, the only potential pole of $E_{\mathfrak{a}}^k(z,w)$ is at $w=1$. This is shown in general in [Iwa97], and the calculations are very similar to those in [Gol06, Chapter 3]. For completeness and as a unifying reference, we recompute and state the exact coefficients of E_∞^k in §2.2.1.

For half-integral weight, the coefficients have similar shape,

$$\begin{aligned}\rho_{\mathfrak{a}}^k(0,w) &= (*)\frac{L_{\mathfrak{a}}^k(4w-2)}{L_{\mathfrak{a}}^k(4w-1)}\frac{\Gamma(2w-1)}{\Gamma(w+\frac{k}{2})\Gamma(w-\frac{k}{2})}\\ \rho_{\mathfrak{a}}^k(h,w) &= (*)|h|^{w-1}\frac{1}{\Gamma(w+\frac{|h|}{h}\frac{k}{2})}D_{\mathfrak{a}}^k(h,w),\end{aligned} \tag{2.1}$$

except that now $D^k_{\mathfrak{a}}(h,w)$ is a complete Dirichlet series formed from Gauss sums, and which factors as

$$D^k_{\mathfrak{a}}(h,w) = \frac{L(2w-\frac{1}{2},\chi_{k,h})}{\zeta(4w-1)}\widetilde{D}_{\mathfrak{a}}(h,w), \tag{2.2}$$

where $\widetilde{D}_{\mathfrak{a}}(h,w)$ is a finite Dirichlet polynomial and $\chi_{k,h}$ is the real quadratic character associated to the extension $\mathbb{Q}(\sqrt{\mu_k h})$, and where $\mu_k = (-1)^{k-\frac{1}{2}}$. In other words, these are almost the same as the full integral weight Eisenstein series, except with an additional L-function in the numerator, and with slightly different arguments of the involved L-functions. For $\operatorname{Re} w > \frac{1}{2}$, the only potential pole of $E^k_{\mathfrak{a}}(z,w)$ is at $w = \frac{3}{4}$. This is all described in [GH85], including the factorization in (2.2), which is given in [GH85, Corollary 1.3]. For completeness and as a unifying reference, we recompute and state the shape of the coefficients of E^k_∞ in §2.2.2, including a proof of the factorization (2.2).

The rest of this section consists of a complete description of these coefficients. However, the general shapes of the coefficients are sufficient for the rest of this thesis.

2.2.1 Full Integral Weight

Claim 2.2.1. *For $k \geq 1$ a full integer, the Fourier-Whittaker coefficients of $E^k_\infty(z,w)$ are given by*

$$\rho^k_\infty(0,w) = \begin{cases} 0 & \text{if } k \text{ odd} \\ y^{1-w}2\pi 4^{1-3w}\dfrac{\zeta(2w-1)}{\zeta^{(2)}(2w)}\dfrac{\Gamma(2w-1)}{\Gamma(w+\frac{k}{2})\Gamma(w-\frac{k}{2})} & \text{if } k \text{ even} \end{cases}$$

$$\rho^k_\infty(h,w) = \frac{\pi^w|h|^{w-1}e^{-\frac{i\pi k}{2}}}{L\big(2w,\big(\frac{-4}{\cdot}\big)^k\big)\Gamma(w+\frac{|h|}{h}\frac{k}{2})}D^k_\infty(h,w)$$

where

$$D^k_\infty(h,w) = \sum_{c|h}\frac{c}{(4c)^{2w}}\left(e^{\frac{\pi i h}{2c}} + (-1)^k e^{\frac{3\pi i h}{2c}}\right)$$

is a finite Dirichlet sum.

This is a classical computation. For completeness and ease of reference, we go through this computation completely. The coefficients for other full integral weight Eisenstein coefficients are very similar, and are achieved through (essentially) the same work.

Proof. In the expression for $E_\infty^k(z,w)$, when $(c,d)=(0,1)$, there is the single term y^w. We compute the rest of the 0th Fourier-Whittaker coefficient through

$$\rho_\infty^k(0,w)=\int_0^1 E_\infty^k(z,w)-y^w dx=\sum_{\substack{c>0,d\in\mathbb{Z}\\ \gcd(4c,d)=1}}\int_0^1 \frac{y^s\left(\frac{-4}{d}\right)^k}{|4cz+d|^{2w-k}(4cz+d)^k}.$$

We write $\left(\frac{-4}{d}\right)$ in place of $\left(\frac{-1}{d}\right)$ to facilitate our next step, and as a way of reinforcing that $\gcd(d,4)=1$. Multiplying through by $L\big(2w,\left(\frac{-4}{\cdot}\right)^k\big)$ and distributing allows us to remove the $\gcd(4c,d)=1$ condition from the sum, so that

$$\begin{aligned}\rho_\infty^k(0,w)&=\frac{y^w}{L\big(2w,\left(\frac{-4}{\cdot}\right)^k\big)}\sum_{c>0}\sum_d\left(\frac{-4}{d}\right)^k\int_0^1\frac{1}{|4cz+d|^{2w-k}(4cz+d)^k}dx\\&=\frac{y^w}{L\big(2w,\left(\frac{-4}{\cdot}\right)^k\big)}\sum_{c>0}\frac{1}{(4c)^{2w}}\sum_d\left(\frac{-4}{d}\right)^k\int_{\frac{d}{4c}}^{1+\frac{d}{4c}}\frac{1}{|z|^{2w-k}z^k}dx.\end{aligned}$$

We can write $d=d'+4cq$ for each d' mod $4c$ and $q\in\mathbb{Z}$ uniquely, and executing the resulting q sum tiles the integral to the whole real line, giving

$$\rho_\infty^k(0,w)=\frac{y^w}{L\big(2w,\left(\frac{-4}{\cdot}\right)^k\big)}\sum_{c>0}\frac{1}{(4c)^{2w}}\sum_{d \bmod 4c}\left(\frac{-4}{d}\right)^k\int_{-\infty}^{\infty}\frac{1}{|z|^{2w-k}z^k}dx.$$

Notice that the d sum is 0 if k is odd, and is $2c$ (the number of odd integers up to $4c$) when k is even. Further, when k is even the L-function in the denominator simplifies to a zeta function missing its 2-factor.

It remains necessary to compute the integral. By [Iwa97, Section 13.7], we have the classical integral transform

$$\int_{-\infty}^{\infty}\frac{1}{|z|^{2w-k}z^k}=y^{1-2w}\pi 4^{1-w}\frac{\Gamma(2w-1)}{\Gamma(w+\frac{k}{2})\Gamma(w-\frac{k}{2})}. \tag{2.3}$$

Therefore, we have for k even that

$$\rho_\infty^k(0,w)=y^{1-w}2\pi 4^{1-3w}\frac{\zeta(2w-1)}{\zeta^{(2)}(2w)}\frac{\Gamma(2w-1)}{\Gamma(w+\frac{k}{2})\Gamma(w-\frac{k}{2})}$$

For the hth Fourier coefficient, the method begins similarly. Following the same initial steps, we compute

$$\begin{aligned}\rho_\infty^k(h,w)&=\int_0^1 E_\infty^k(z,w)e^{-2\pi ihx}dx\\&=\frac{y^w}{L(2w,\left(\frac{-4}{\cdot}\right)^k)}\sum_{c>0}\frac{1}{(4c)^{2w}}\sum_d\left(\frac{-4}{d}\right)^k e^{2\pi i\frac{hd}{4c}}\int_{\frac{d}{4c}}^{1+\frac{d}{4c}}\frac{e^{-2\pi ihx}}{|z|^{2w-k}z^k}dx.\end{aligned}$$

Writing $d = d' + 4cq$ as above again tiles out the integral, so that

$$\rho_\infty^k(h,w) = \frac{y^w}{L(2w, \left(\frac{-4}{\cdot}\right)^k)} \sum_{c>0} \frac{1}{(4c)^{2w}} \sum_{d \bmod 4c} \left(\frac{-4}{d}\right)^k e^{2\pi i \frac{hd}{4c}} \int_{-\infty}^{\infty} \frac{e^{-2\pi i h x}}{|z|^{2w-k} z^k} dx.$$

For $h \neq 0$, the integral can be evaluated as in [Iwa97, §13.7] or using [JZ07, 3.385.9] to be

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i h x}}{|z|^{2w-k} z^k} dx = \frac{y^{-w} i^{-k} \pi^w |h|^{w-1}}{\Gamma(w + \frac{|h|}{h}\frac{k}{2})} W_{\frac{|h|}{h}\frac{k}{2}, \frac{1}{2}-w}(4\pi|h|y). \tag{2.4}$$

To understand the arithmetic part,

$$\sum_{c>0} \frac{1}{(4c)^{2w}} \sum_{d \bmod 4c} \left(\frac{-4}{d}\right)^k e^{\pi i \frac{hd}{2c}},$$

note that each d can be written uniquely in the form $d = d' + 4q$ for $0 \leq d' < 4$ and $0 \leq q < c$. Then the arithmetic part is written

$$\sum_{c>0} \frac{1}{(4c)^{2w}} \sum_{d=0}^{3} \left(\frac{-4}{d}\right)^k e^{\pi i \frac{hd}{2c}} \sum_{q \bmod c} e^{2\pi i \frac{hq}{c}}.$$

The final sum over q is

$$\sum_{q \bmod c} e^{2\pi i \frac{hq}{c}} = \begin{cases} 0 & \text{if } c \nmid h \\ c & \text{if } c \mid h. \end{cases}$$

Therefore the arithmetic part simplifies down to

$$\sum_{c|h} \frac{c}{(4c)^{2w}} \left(e^{\pi i \frac{h}{2c}} + (-1)^k e^{\frac{3\pi i h}{2c}}\right).$$

Simplification completes the proof. □

2.2.2 Half Integral Weight

The pattern here is almost the exact same as with full-integral weight. We will prove the expansion for $E_\infty^k(z,w)$ completely. The primary difference is that there is now a Dirichlet series of Gauss sums, which can be factored as a ratio of a Dirichlet L-function and a zeta function, up to a short correction factor.

Claim 2.2.2. *For $k \geq \frac{1}{2}$ a half integer, the Fourier-Whittaker coefficients of $E_\infty^k(z,w)$ is given by*

$$\rho_\infty^k(0,w) = \left(\frac{1+i^{2k}}{2}\right) \frac{\pi 4^{1-w}}{2^{4w-1}-1} \frac{\zeta(4w-2)}{\zeta(4w-1)} \frac{\Gamma(2w-1)}{\Gamma(w+\frac{k}{2})\Gamma(w-\frac{k}{2})}$$

$$\rho_\infty^k(h,w) = \frac{e^{\frac{-i\pi k}{2}}\pi^w|h|^{w-1}}{\Gamma(w+\frac{|h|}{h}\frac{k}{2})}\sum_{c>0}\frac{g_h(4c)}{(4c^{2w})},$$

where

$$g_h(c) = \sum_{d \bmod c} \varepsilon_d^{2k}\left(\frac{c}{d}\right)e\big(\tfrac{hd}{c}\big)$$

is a Gauss sum. The Dirichlet series associated to these Gauss sums can be written as

$$\sum_{c\geq 1}\frac{g_h(4c)}{(4c)^{2w}} = \frac{L^{(2)}(2w-\frac{1}{2},\chi_{k,h})}{\zeta^{(2h)}(4w-1)}\widetilde{D}_\infty^k(h,w),$$

as proved in Proposition 2.2.3.

Proof. As in the full-integral weight case, we compute the constant Fourier coefficient through

$$\rho_\infty^k(0,w) = \int_0^1 E_\infty^k(z,w) - y^w dx = \sum_{\substack{c>0,d\in\mathbb{Z}\\ \gcd(4c,d)=1}}\int_0^1 \frac{y^w\varepsilon_d^{2k}\left(\frac{4c}{d}\right)}{|4cz+d|^{2w-k}(4cz+d)^k}.$$

The only difference in comparison to the full-integral weight case is that the numerator has $\varepsilon_d^{2k}\left(\frac{4c}{d}\right)$. Notice that the character enforces $\gcd(4c,d)=1$, so that this condition can be dropped from the summation indices.

Factoring $(4c)^{-2w}$, writing $d = d' + 4cq$ for each $d' \bmod 4c$ and a unique $q\in\mathbb{Z}$, performing the change of variables $x \mapsto x - \frac{d}{4c}$, and tiling the integral functions exactly as in the full-integral proof, and leads to

$$y^w\sum_{c>0}\frac{1}{(4c)^{2w}}\sum_{d \bmod 4c}\varepsilon_d^{2k}\left(\frac{4c}{d}\right)\int_{-\infty}^{\infty}\frac{1}{|z|^{2w-k}z^k}dx.$$

Using the evaluation of this integral from (2.3), we see that the 0th coefficient can be written as

$$y^{1-w}\sum_{c>0}\frac{1}{(4c)^{2w}}\sum_{d \bmod 4c}\varepsilon_d^{2k}\left(\frac{4c}{d}\right)\frac{\pi 4^{1-w}\Gamma(2s-1)}{\Gamma(w+\frac{k}{2})\Gamma(w-\frac{k}{k})}.$$

It is now necessary to understand the arithmetic part, given by

$$\sum_{c>0}\frac{1}{(4c)^{2w}}\sum_{d \bmod 4c}\varepsilon_d^{2k}\left(\frac{4c}{d}\right). \tag{2.5}$$

Noting that we can rewrite ε_d as

$$\varepsilon_d = \frac{1}{2}\big(\chi_{-1}^2(d)+\chi_{-1}(d)\big) + \frac{i}{2}\big(\chi_{-1}^2(d)-\chi_{-1}(d)\big),$$

where $\chi_{-1}(d) = \left(\frac{-1}{d}\right)$ is the Legendre character associated to when -1 is a square, we see that the d summation is trivial unless the character $\left(\frac{4c}{d}\right)$ is trivial. (This follows from classical observation that summing a non-trivial character over a whole group gives zero). Thus the sum is nontrivial if and only if c is a square, and when c is a square we have that

$$\sum_{d \bmod 4c^2} \varepsilon_d^{2k} \left(\frac{4c^2}{d}\right) = \sum_{\substack{d \bmod 4c^2 \\ \gcd(d,4c)=1}} \varepsilon_d^{2k} = \frac{1+i^{2k}}{2}\varphi(4c^2).$$

Therefore (2.5) can be written as

$$\frac{1+i^{2k}}{2}\sum_{c>0}\frac{1}{(4c^2)^{2w}}\varphi(4c^2) = \frac{1+i^{2k}}{2}\frac{1}{2^{4w-1}-1}\frac{\zeta(4w-2)}{\zeta(4w-1)}.$$

The last equality follows from comparing Euler products, correcting the missing 2-factor, and simplifying. Combining the arithmetic part with the analytic part gives the $\rho_\infty^k(0,w)$.

Computing the hth Fourier coefficient is very similar. After following the same initial steps to tile out the integral, we get the equality

$$\int_0^1 E_\infty^k(z,w)e^{-2\pi ihx}dx = \sum_{c>0}\frac{1}{(4c)^{2w}}\sum_{d \bmod 4c}\varepsilon_d^{2k}\left(\frac{4c}{d}\right)e\left(\tfrac{hd}{4c}\right)\int_{-\infty}^{\infty}\frac{y^w e^{-2\pi ihx}}{|z|^{2w-k}z^k}dx.$$

This integral was evaluated in (2.4), so we see that this becomes

$$\frac{e^{-i\pi k/2}\pi^w|h|^{w-1}}{\Gamma(w+\frac{|h|}{h}\frac{k}{2})}W_{\frac{|h|}{h}\frac{k}{2},\frac{1}{2}-w}(4\pi|h|y)\sum_{c>0}\frac{1}{(4c)^{2w}}\sum_{d \bmod 4c}\varepsilon_d^{2k}\left(\frac{4c}{d}\right)e\left(\tfrac{hd}{4c}\right).$$

The arithmetic part is now a Dirichlet series of Gauss sums, and is not finite. Simplifiying completes the computation. □

On the L-series associated to half-integral weight Eisenstein series coefficients

We now consider the Dirichlet series

$$\sum_{c\geq 1}\frac{g_h(4c)}{(4c)^{2w}}$$

appearing in the hth Fourier coefficients of $E_\infty^k(z,w)$ when k is a half integer. The goal of this section is to show the following proposition.

Proposition 2.2.3.

$$\sum_{c\geq 1}\frac{g_h(4c)}{(4c)^{2w}}=\frac{L^{(2)}(2w-\frac{1}{2},\chi_{k,h})}{\zeta^{(2h)}(4w-1)}\widetilde{D}^k_\infty(h,w),$$

where $\widetilde{D}^k_\infty(h,w)$ is a finite Dirichlet polynomial.

We prove this proposition through a series of lemmata.[†] First, we first consider just an individual Gauss sum $g_h(4c)=\sum_{d \bmod 4c}\varepsilon_d^{2k}e(\frac{hd}{4c})\left(\frac{4c}{d}\right)$. It is necessary to break each Gauss sum into two pieces.

Lemma 2.2.4.

$$g_h(4c)=\sum_{d_2 \bmod 2^\alpha}\varepsilon_{d_2c'}^{2k}(-1)^{\frac{c'-1}{2}\frac{d_2c'-1}{2}}\varepsilon_{c'}^{-1}\left(\frac{2^\alpha}{d_2}\right)\varepsilon_{c'}\sum_{d_1 \bmod c'}\left(\frac{d_1}{c'}\right)e\left(\tfrac{hd_1}{c'}\right).$$

Proof. Write $4c=2^\alpha c'$ where $\gcd(c',2)=1$. Note that we necessarily have $\alpha\geq 2$. For any $d \bmod 4c$, we write $d=d_1 2^\alpha+d_2c'$ with d_1 varying mod c' and d_2 varying mod 2^α. Using this, we can write

$$\begin{aligned}\sum_{d \bmod 4c}\varepsilon_d^{2k}e\left(\tfrac{hd}{4c}\right)\left(\frac{4c}{d}\right)&=\sum_{(d_1 \bmod c')}\sum_{(d_2 \bmod 2^\alpha)}\varepsilon_{d_2c'}^{2k}e\left(\tfrac{h(d_1 2^\alpha+d_2c')}{2^\alpha c'}\right)\left(\frac{2^\alpha c'}{d_1 2^\alpha+d_2c'}\right)\\&=\sum_{d_2 \bmod 2^\alpha}\varepsilon_{d_2c'}^{2k}e\left(\tfrac{hd_2}{2^\alpha}\right)\left(\frac{2^\alpha}{d_2c'}\right)\sum_{d_1 \bmod c'}\left(\frac{c'}{d_1 2^\alpha+d_2c'}\right)e\left(\tfrac{hd_1}{c'}\right).\end{aligned}$$

Note that d_2 is necessarily odd. By quadratic reciprocity, we can rewrite the inner sum as

$$\begin{aligned}\sum_{d_1 \bmod c'}\left(\frac{c'}{d_1 2^\alpha+d_2c'}\right)e\left(\tfrac{hd_1}{c'}\right)&=\sum_{d_1 \bmod c'}\left(\frac{d_1 2^\alpha+d_2c'}{c'}\right)(-1)^{\frac{c'-1}{2}\frac{d_2c'-1}{2}}e\left(\tfrac{hd_1}{c'}\right)\\&=\sum_{d_1 \bmod c'}\left(\frac{d_1 2^\alpha}{c'}\right)(-1)^{\frac{c'-1}{2}\frac{d_2c'-1}{2}}e\left(\tfrac{hd_1}{c'}\right).\end{aligned}$$

Inserting this back into $g_h(4c)$ gives

$$\begin{aligned}g_h(4c)&=\sum_{d_2 \bmod 2^\alpha}\varepsilon_{d_2c'}^{2k}(-1)^{\frac{c'-1}{2}\frac{d_2c'-1}{2}}e\left(\tfrac{hd_2}{2^\alpha}\right)\left(\frac{2^\alpha}{d_2}\right)\left(\frac{2^\alpha}{c'}\right)\sum_{d_1 \bmod c'}\left(\frac{2^\alpha}{c'}\right)\left(\frac{d_1}{c'}\right)e\left(\tfrac{hd_1}{c'}\right)\\&=\sum_{d_2 \bmod 2^\alpha}\varepsilon_{d_2c'}^{2k}(-1)^{\frac{c'-1}{2}\frac{d_2c'-1}{2}}e\left(\tfrac{hd_2}{2^\alpha}\right)\varepsilon_{c'}^{-1}\left(\frac{2^\alpha}{d_2}\right)\varepsilon_{c'}\sum_{d_1 \bmod c'}\left(\frac{d_1}{c'}\right)e\left(\tfrac{hd_1}{c'}\right).\end{aligned}$$

Note that we have introduced $\varepsilon_c\varepsilon_c^{-1}$. This ends the proof of the lemma. □

[†]Yes, that is a real word. And it's a fantastic word.

The d_1 sum is now completely decoupled from the d_2 sum, and they can be understood separately. We first understand the d_1 summation.

Lemma 2.2.5. *For odd c, define*

$$H_h(c) := \varepsilon_c \sum_{d \bmod c} \left(\frac{d}{c}\right) e\big(\tfrac{hd}{c}\big),$$

as occurs in the d_1 summation in $g_h(4c)$. Then $H_h(c)$ is multiplicative.

Proof. Let n_1 and n_2 be two odd relative prime integers. By the Chinese Remainder Theorem, and congruence class d mod n_1n_2 can be uniquely written as $d = b_2n_1+b_1n_2$ where b_2 is defined modulo n_2 and b_1 is defined modulo n_1. Then

$$\begin{aligned}
H_h(n_1n_2) &= \varepsilon_{n_1n_2} \sum_{d \bmod n_1n_2} \left(\frac{d}{n_1n_2}\right) e\big(\tfrac{dh}{n_1n_2}\big) \\
&= \varepsilon_{n_1n_2} \sum_{b_1 \bmod n_1} \sum_{b_2 \bmod n_2} \left(\frac{b_2n_1 + b_1n_2}{n_1n_2}\right) e\big(\tfrac{hb_2n_1+b_1n_2}{n_1n_2}\big) \\
&= \varepsilon_{n_1n_2} \left(\frac{n_2}{n_1}\right)\left(\frac{n_1}{n_2}\right) \sum_{b_1 \bmod n_1} \left(\frac{b_1}{n_1}\right) e\big(\tfrac{hb_1}{n_1}\big) \sum_{b_2 \bmod n_2} \left(\frac{b_2}{n_2}\right) e\big(\tfrac{hb_2}{n_2}\big) \\
&= \varepsilon_{n_1n_2}\varepsilon_{n_1}^{-1}\varepsilon_{n_2}^{-1} \left(\frac{n_2}{n_1}\right)\left(\frac{n_1}{n_2}\right) H_h(n_1)H_h(n_2).
\end{aligned}$$

Casework and quadratic reciprocity shows that $\varepsilon_{n_1n_2}\varepsilon_{n_1^{-1}}\varepsilon_{n_2}^{-1}\left(\frac{n_2}{n_1}\right)\left(\frac{n_1}{n_2}\right) = 1$, so that $H_h(n_1n_2) = H_h(n_1)H_h(n_2)$. □

In order to understand a Dirichlet series formed from $H_h(c')$, it will be sufficient to understand $H_h(p^k)$ for odd primes p. When $p \nmid h$, these are particularly easy to understand.

Lemma 2.2.6. *Suppose p is an odd prime and $p \nmid h$. Then*

$$H_h(p^k) = \begin{cases} \left(\frac{-h}{p}\right)\sqrt{p} & k = 1 \\ 0 & k \geq 2. \end{cases}$$

Proof. When $k = 1$, we have

$$H_h(p) = \varepsilon_p \sum_{d \bmod p} \left(\frac{d}{p}\right) e\big(\tfrac{dh}{p}\big) = \varepsilon_p^2 \left(\frac{h}{p}\right)\sqrt{p} = \left(\frac{-h}{p}\right)\sqrt{p},$$

as $H_h(p)$ is very nearly a standard Gauss Sum, as considered in the beginning of [Dav80].

For $k \geq 2$, there are two cases. If k is even, then the sum is exactly a sum over the primitive p^k-roots of unity, and therefore is zero. If k is odd, then writing $\eta = e^{2\pi i/p^k}$ as a primitive p^k-root of unity, we have

$$H_h(p^k) = \sum_{d \bmod p^k} \eta^d \left(\frac{d}{p^k}\right) = \sum_{d \bmod p^k} \eta^d \left(\frac{d}{p}\right) = \sum_{\substack{b \bmod p \\ c \bmod p^{k-1}}} \eta^{b+pc} \left(\frac{b}{p}\right)$$
$$= \sum_{b \bmod p} \eta^b \left(\frac{b}{p}\right) \sum_{c \bmod p^{k-1}} \eta^{pc} = 0,$$

as the inner sum is a sum over the primitive p^{k-1} roots of unity. □

The case is substantially more complicated for primes p dividing h. For this thesis, we do not need to calculate these explicitly. (Although we could, using these techniques). It is sufficient to know that only finitely many contribute.

Lemma 2.2.7. *Suppose p is an odd prime and $p \mid h$. Further, suppose $p^\ell \mid h$ but $p^{\ell+1} \nmid h$. Then for $k \geq \ell + 2$, we have $H_h(p^k) = 0$.*

Proof. This is substantially similar to the case when $p \nmid h$, and the same proof carries over. When k is even, $H_h(p^k)$ is a sum over primitive $p^{k-\ell}$ roots of unity. When k is odd, $H_h(p^k)$ has an inner exponential sum over the $p^{k-\ell-1}$ roots of unity. □

Remark 2.2.8. Although we do not compute it here, it is possible to compute the exact contribution from each factor p dividing h. One complete reference can be found at the author's website [LD]. Some subcases are included in [GH85].

We now seek to understand the d_2 summation in Lemma 2.2.4.

Lemma 2.2.9. *We have*

$$\varepsilon_{d_2c'}^{2k}(-1)^{\frac{c'-1}{2}\frac{d_2c'-1}{2}}\varepsilon_{c'}^{-1} = \varepsilon_{d_2}^{2k}\left(\frac{(-1)^{k+\frac{1}{2}}}{c'}\right). \tag{2.6}$$

In particular,

$$\sum_{d_2 \bmod 2^\alpha} \varepsilon_{d_2c'}^{2k}(-1)^{\frac{c'-1}{2}\frac{d_2c'-1}{2}} e\left(\tfrac{hd_2}{2^\alpha}\right)\varepsilon_{c'}^{-1}\left(\frac{2^\alpha}{d_2}\right) = \sum_{d_2 \bmod 2^\alpha} \varepsilon_{d_2}^{2k}\left(\frac{(-1)^{k+\frac{1}{2}}}{c'}\right)\left(\frac{2^\alpha}{d_2}\right) e\left(\tfrac{hd_2}{2^\alpha}\right).$$

Proof. First note that as $\varepsilon_d^4 = 1$, we can reduce the analysis into two cases: when $2k \equiv 1 \bmod 4$ and when $2k \equiv 3 \bmod 4$. After this reduction, the equality in (2.6) is quickly verified by considering the possible values of d_2 and c' modulo 4 (recalling that both d_2 and c' are odd). The rest of the lemma follows immediately. □

In this simplified d_2 summation, there appears $\varepsilon_{d_2}^{2k}$, which acts a bit like a character in d_2 modulo 4, and $\left(\frac{2^\alpha}{d_2}\right)$, which acts a bit like a character modulo 8, and an unrestrained exponential. Therefore we should expect that when $2^\alpha > 8$, or rather when $\alpha \geq 4$, then the entire d_2 sum vanishes unless h is highly divisible by 2.

Lemma 2.2.10. *Suppose $2^\ell \mid h$ and $2^{\ell+1} \nmid h$. If $\alpha \geq \ell + 4$, then the d_2 summation vanishes, i.e.*

$$\sum_{d_2 \bmod 2^\alpha} \varepsilon_{d_2}^{2k} \left(\frac{2^\alpha}{d_2}\right) e\big(\tfrac{hd_2}{2^\alpha}\big) = 0.$$

Proof. Without loss of generality, choose least non-negative representations for each $d_2 \bmod 2^\alpha$. Write $d_2 = 8d' + d''$ where $0 \leq d' < 2^{\alpha-3}$ and $0 \leq d'' < 8$. Then $\varepsilon_{8d'+d''} = \varepsilon_{d''}$ and $\left(\frac{2^\alpha}{8d'+d''}\right) = \left(\frac{2^\alpha}{d''}\right)$, so the d' summation can be considered separately. This summation is

$$\sum_{d'=0}^{2^{\alpha-3}-1} e\left(\frac{hd'}{2^{\alpha-3}}\right),$$

which is 0 unless $2^{\alpha-3} \mid h$. □

As $c = 2^\alpha c'$, we see that the d_2 summation constrains the contribution from the 2-factor of c. We are now finally ready to prove Proposition 2.2.3.

Proof of Prop 2.2.3. We try to understand

$$\begin{aligned}
&\sum_{c>0} \frac{1}{(4c)^{2w}} \sum_{d \bmod 4c} \varepsilon_d^{2k} e\big(\tfrac{hd}{4c}\big) \left(\frac{4c}{d}\right) \\
&= \sum_{\alpha \geq 2} \sum_{\substack{c \geq 1 \\ \gcd(2,c)=1}} \frac{1}{(2^\alpha c)^{2w}} \left(\frac{(-1)^{k+\frac{1}{2}}}{c}\right) H_h(c) \sum_{d_2 \bmod 2^\alpha} \varepsilon_{d_2}^{2k} e\big(\tfrac{d_2 h}{2^\alpha}\big) \left(\frac{2^\alpha}{d_2}\right) \\
&= \left(\sum_{\alpha \geq 2} \sum_{d_2 \bmod 2^\alpha} \frac{1}{2^{2\alpha w}} \varepsilon_{d_2}^{2k} e\big(\tfrac{d_2 h}{2^\alpha}\big) \left(\frac{2^\alpha}{d_2}\right)\right) \left(\sum_{\substack{c \geq 1 \\ \gcd(c,2)=1}} \frac{1}{c^{2w}} \Big(\frac{(-1)^{k+\frac{1}{2}}}{c}\Big) H_h(c)\right). \qquad (2.7)
\end{aligned}$$

We have used Lemma 2.2.4 and Lemma 2.2.9 to split and simplify this expression. Let $\chi_k(c) := \left(\frac{(-1)^{k+\frac{1}{2}}}{c}\right)$ for ease of notation. As the summands over c are multiplicative (by Lemma 2.2.5), the c sum can be written (for $\operatorname{Re} w \gg 1$) as

$$\sum_{\substack{c \geq 1 \\ \gcd(c,2)=1}} \frac{1}{c^{2w}} \chi_k(c) H_h(c) = \prod_{\substack{p \\ p \neq 2}} \Big(1 + \frac{\chi_k(p) H_h(p)}{p^{2w}} + \frac{\chi_k(p^2) H_h(p^2)}{p^{4w}} + \cdots\Big).$$

For primes not dividing h, this expression simplifies significantly as $H_h(p^k) = 0$ for $k \geq 2$ (as shown in Lemma 2.2.6). The product over these primes is then

$$\prod_{\substack{p \\ p \neq 2 \\ p \nmid h}} \Big(1 + \frac{\big(\frac{h(-1)^{k-\frac{1}{2}}}{p}\big)}{p^{2w-\frac{1}{2}}}\Big).$$

On primes avoiding h and 2, it's quickly checked that this perfectly matches the Euler product for $L(2w-\frac{1}{2}, \chi_{k,h})\zeta(4w-1)^{-1}$, where $\chi_{k,h}(\cdot) = \big(\frac{h(-1)^{k-\frac{1}{2}}}{\cdot}\big)$. Therefore we can write the product over those primes avoiding h and 2 as

$$\prod_{\substack{p \\ p \neq 2 \\ p \nmid h}} \Big(1 + \frac{\big(\frac{h(-1)^{k-\frac{1}{2}}}{p}\big)}{p^{2w-\frac{1}{2}}}\Big) = \frac{L^{(2)}(2w-\frac{1}{2}, \chi_{k,h})}{\zeta^{(2h)}(4w-1)}.$$

(Note that we are using the convention that $L^{(Q)}(s)$ denotes an L-function $L(s)$, but with the Euler factors for primes p dividing Q removed).

We now consider the primes p which do divide h in the Euler product in (2.7). By Lemma 2.2.7, we see that $H_h(p^k) = 0$ when $p^{k-1} \nmid h$. Therefore the product over primes dividing h is a product of finitely many terms of finite length, and is thus just a Dirichlet polynomial.

Similarly, by Lemma 2.2.10, the sum over d_2 and α in (2.7) is a finite sum whose length depends on the 2-factor of h, and is also a Dirichlet polynomial.

We group the product over primes dividing h with the d_2 and α summation in (2.7), which are both finite Dirichlet polynomials, into a single Dirichlet polynomial

$$\widetilde{D}^k_\infty(h,w) := \sum_{\alpha \geq 2} \sum_{d_2 \bmod 2^\alpha} \frac{1}{2^{2\alpha w}} \varepsilon_{d_2}^{2k} e\big(\tfrac{d_2 h}{2^\alpha}\big) \left(\frac{2^\alpha}{d_2}\right) \times \prod_{\substack{p \\ p|h \\ p \neq 2}} \sum_{j \geq 0} \frac{\chi_k(p^j) H_h(p^j)}{p^{2jw}}.$$

Note carefully that although this is written as an infinite polynomial, it is a finite Dirichlet polynomial.

Collecting these pieces together we have now shown that

$$\sum_{c>0} \frac{1}{(4c)^{2w}} \sum_{d \bmod 4c} \varepsilon_d^{2k} e\big(\tfrac{hd}{4c}\big) \left(\frac{4c}{d}\right) = \frac{L^{(2)}(2w-\frac{1}{2}, \chi_{k,h})}{\zeta^{(2h)}(4w-1)} \widetilde{D}^k_\infty(h,w),$$

as we set out to show. □

Remark 2.2.11. If h is squarefree, then it is possible to show that $\widetilde{D}^k_\infty(h,w)$ has the necessary Euler factors to "fill in" the h factors of $\zeta^{(2h)}(4w-q)$ (although not the 2 factor), and the expression for $\widetilde{D}^k_\infty(h,w)$ simplifies significantly. Thus the case when h is squarefree is significantly simpler.

2.3 Cutoff Integrals and Their Properties

We recall the Mellin transform and inverse Mellin transform, and use these to construct appropriate integral transforms to analyze properties of the coefficients of a Dirichlet series. In general, if

$$F(s) = \int_0^\infty f(x)x^s \frac{dx}{x},$$

then $F(s)$ is the *Mellin transorm* of $f(x)$. Mellin transforms are deeply related to Laplace transforms and Fourier transforms, and when f and F are sufficiently nice, there is an analogous inversion theorem giving that

$$f(x) = \frac{1}{2\pi i}\int_{(\sigma)} F(s)x^{-s}ds.$$

Cesàro Cutoff Transform

In this thesis, we reintroduce and use Cesàro weights. Note that these are sometimes referred to as "Riesz Means." Given a positive integer k and a Dirichlet series

$$D(s) = \sum_{n\geq 1}\frac{a(n)}{n^s},$$

we have the fundamental relationship

$$\begin{aligned}\frac{1}{k!}\sum_{n\leq X} a(n)\Big(1-\frac{n}{X}\Big)^k &= \frac{1}{2\pi i}\int_{(\sigma)} D(s)\frac{X^s}{s(s+1)\cdots(s+k)}ds \\ &= \frac{1}{2\pi i}\int_{(\sigma)} D(s)\frac{X^s\Gamma(s)}{\Gamma(s+k+1)}ds,\end{aligned} \tag{3.1}$$

where σ is large enough that $D(s)$ and the integral absolutely converge. The individual weights $(1-\frac{n}{X})^k$ on each $a(n)$ are the k-Cesàro weights, and give access to smoothed asymptotics.

The relationship (3.1) follows from the following integral equality.

Lemma 2.3.1. *For $\sigma > 0$, we have*

$$\frac{1}{2\pi i}\int_{(\sigma)} \frac{Y^s}{s(s+1)\cdots(s+k)}ds = \begin{cases} \frac{1}{k!}\big(1-\frac{1}{Y}\big)^k & \text{if } Y \geq 1, \\ 0 & \text{if } Y < 1. \end{cases}$$

Proof. In the case when $Y \geq 1$, shifting the contour infinitely far to the left shows that the integral can be evaluated as

$$\sum_{j=0}^{k} \operatorname*{Res}_{s=-j}\left(\frac{Y^s}{s(s+1)\cdots(s+k)}\right) = \sum_{j=0}^{k} \frac{(-1)^j Y^{-j}}{j!(k-j)!} = \frac{1}{k!}\Big(1-\frac{1}{Y}\Big)^k.$$

Note that the last equality is an application of the binomial theorem.

In the case when $Y < 1$, shifting the contour infinitely far to the right shows that the integral is 0. □

To recover (3.1), one expands $D(s)$ within the integral and applies the lemma to each individual term with $Y = (X/n)$.

Exponentially Smoothed Integral

The integral definition of the Gamma function

$$\Gamma(s) = \int_0^\infty t^s e^{-t}\frac{dt}{t}$$

is a Mellin integral, and gives the inverse Mellin integral

$$e^{-x} = \frac{1}{2\pi i}\int_{(\sigma)} x^{-s}\Gamma(s)ds.$$

Applied to the Dirichlet series $D(s)$, we have

$$\sum_{n\geq 1} a(n)e^{-n/X} = \frac{1}{2\pi i}\int_{(\sigma)} D(s)X^s\Gamma(s)ds.$$

Concentrating Integral

We now produce an integral transform that has the effect of concentrating the mass of the integral around the parameter X. We claim that

$$\frac{1}{2\pi i}\int_{(2)} \exp\left(\frac{\pi s^2}{Y^2}\right)\frac{X^s}{Y}ds = \frac{1}{2\pi}\exp\left(-\frac{Y^2\log^2 X}{4\pi}\right).$$

Proof. Write $X^s = e^{s \log X}$ and complete the square in the exponents. Since the integrand is entire and the integral is absolutely convergent, performing the change of variables $s \mapsto s - Y^2 \log X/2\pi$ and shifting the line of integration back to the imaginary axis yields

$$\frac{1}{2\pi i} \exp\left(-\frac{Y^2 \log^2 X}{4\pi}\right) \int_{(0)} e^{\pi s^2/Y^2} \frac{ds}{Y}.$$

The change of variables $s \mapsto isY$ transforms the integral into the standard Gaussian, completing the proof. □

Applied to a Dirichlet series $D(s)$, we have

$$\frac{1}{2\pi} \sum_{n \geq 1} a(n) \exp\left(-\frac{Y^2 \log^2(X/n)}{4\pi}\right) = \frac{1}{2\pi i} \int_{(\sigma)} \exp\left(\frac{\pi s^2}{Y^s}\right) \frac{X^s}{Y} ds.$$

Note in particular that when $|n - X|$ is large, there is significant exponential decay. Therefore this integral concentrates the mass of the expression very near X (and in particular in an interval of width X/Y around X).

Cutoff Integrals from Smooth, Compactly Supported Functions

It will also be useful to document a more general family of cutoff transforms. For $X, Y > 0$, let $\phi_Y(X)$ denote a smooth non-negative function with maximum value 1, satisfying

(i) $\phi_Y(X) = 1$ for $X \leq 1$,

(ii) $\phi_Y(X) = 0$ for $X \geq 1 + \frac{1}{Y}$.

Let $\Phi_Y(s)$ denote the Mellin transform of $\phi_Y(x)$, given by

$$\Phi_Y(s) = \int_0^\infty t^s \phi_Y(t) \frac{dt}{t},$$

defined initially for $\operatorname{Re} s > 0$. Repeated applications of integration by parts and differentiation under the integral shows that $\Phi_Y(s)$ satisfies the following four properties:

(i) $\Phi_Y(s) = \frac{1}{s} + O_s(\frac{1}{Y})$,

(ii) $\Phi_Y'(s) = -\frac{1}{s^2} + O_s(\frac{1}{Y})$

(iii) $\Phi_Y(s) = -\frac{1}{s}\int_1^{1+\frac{1}{Y}} \phi_Y'(t)t^s dt$,

(iv) and for all positive integers m, for s constrained within a vertical strip and $|s-1| > \epsilon$, we have

$$\Phi_Y(s) \ll_\epsilon \frac{1}{Y}\Big(\frac{Y}{1+|s|}\Big)^m.$$

Further, the last property can be extended to real $m > 1$ through the Phragmén-Lindelöf principle. The Mellin transform pair $\Phi_Y(s), \phi_Y(x)$ gives a general set of integral cutoff relations,

$$\sum_{n\le X} a(n) + \sum_{X<n\le X+X/Y} a(n)\phi_Y\Big(\frac{n}{X}\Big) = \frac{1}{2\pi i}\int_{(\sigma)} D(s)\Phi_Y(s)X^s ds.$$

Chapter Three

On Dirichlet Series for Sums of Coefficients of Cusp Forms

3.1 Introduction

Continuing with same notation as before, let f be a holomorphic cusp form of positive weight k on a congruence subgroup $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$, where $k \in \mathbb{Z} \cup (\mathbb{Z} + \frac{1}{2})$. Denote the Fourier expansion of f at ∞ by

$$f(z) = \sum_{n \geq 1} a(n) e(nz).$$

The individual coefficients $a(n)$ have long been of interest since the coefficients contain interesting arithmetic data. For example, the major insight leading to the resolution of Fermat's Last Theorem involved showing that for an elliptic curve E there exists a corresponding modular form f_E whose coefficients (at prime indices) satisfy

$$a(p) = p + 1 - \#E(\mathbb{F}_p),$$

or rather that the $a(p)$ counted the number of points on the elliptic curve over finite fields.

The first cusp form to be studied in depth was the Delta Function (as described in Chapter 1), whose coefficients are the Ramanujan τ function,

$$\Delta(z) = \sum_{n \geq 1} \tau(n) e(nz).$$

Ramanujan conjectured that the coefficients of Δ should satisfy the bound

$$|\tau(n)| \ll d(n) n^{\frac{11}{2}},$$

where $d(n)$ is the number of positive divisors of n. This conjecture initiated an exploration that included a much wider set of objects than Ramanujan could have dreamt of.

Ramanujan's Conjecture has been extended to include all modular and automorphic forms. For a cusp form f of weight k in $\mathrm{GL}(2)$, the conjecture states that

$$a(n) \ll n^{\frac{k-1}{2} + \epsilon}.$$

This is now known as the Ramanujan-Petersson conjecture, and it is now known to be true for full integral weight k holomorphic cusp forms on $\mathrm{GL}(2)$ as a consequence of Deligne's proof of the Weil Conjecture [Del74].

It is an interesting coincidence that Hardy and Littlewood were investigating averaged estimates for the Gauss Circle and Dirichlet Divisor problems when Ramanujan was arriving in England, thinking about the Delta function. It is easy

to look at the first several τ coefficients and believe that the sign of $\tau(n)$ changes approximately uniformly random in n. Under this assumption, and assuming Ramanujan's Conjecture that $\tau(n) \ll n^{\frac{11}{2}+\epsilon}$, it is very natural to conjecture that the summatory function of $\tau(n)$ satisfies the square-root cancellation phenomenon,

$$\sum_{n \leq X} \tau(n) \ll X^{\frac{11}{2}+\frac{1}{2}+\epsilon}.$$

As described in Chapter 1, this is analogous to the error terms $E(R)$ in the Gauss Circle or Dirichlet Divisor Problems. We similarly expect the even better bound,

$$\sum_{n \leq X} \tau(n) \ll X^{\frac{11}{2}+\frac{1}{4}+\epsilon}.$$

Although there is a clear qualitative connection with the Circle and Divisor problems, it seems unlikely that this common thread was recognized by Hardy, Littlewood, or Ramanujan at the time.

For our general cusp form f of weight k in GL(2), we expect an analogous conjecture to hold, which we refer to as the "Classical Conjecture."

Conjecture 3.1.1 (Classical Conjecture)**.** *Let $f(z) = \sum_{n \geq 1} a(n)e(nz)$ be a holomorphic cusp form of weight k on* GL(2)*, where $k \in \mathbb{Z} \cup (\mathbb{Z} + \frac{1}{2})$ and $k > 1$. Then*

$$S_f(n) := \sum_{n \leq X} a(n) \ll X^{\frac{k-1}{2}+\frac{1}{4}+\epsilon}.$$

In a seminal pair of works, Chandrasekharan and Narasimhan [CN62, CN64] showed that the Classical Conjecture is true *on average* by showing that

$$\sum_{n \leq X} |S_f(n)|^2 = CX^{k-1+\frac{3}{2}} + B(X)$$

where $B(X)$ is an error term satisfying

$$B(X) = \begin{cases} O(X^k \log^2 X) \\ \Omega(X^{k-\frac{1}{4}} \frac{(\log\log\log X)^2}{\log X}), \end{cases}$$

and where C is an explicitly known constant.

This should be thought of as a Classical Conjecture *on average* due to the following classical argument:

$$\left(\sum_{n \leq X} |S_f(n)| \right)^2 = \left(\sum_{n \leq X} |S_f(n)| \cdot 1 \right)^2$$

$$\leq \sum_{n\leq X} |S_f(n)|^2 \sum_{m\leq X} 1 = X \sum_{n\leq X} |S_f(n)|^2$$
$$\ll X^{k-1+\frac{5}{2}}.$$

The Cauchy-Schwarz-Bunyakovsky inequality is used to pass from the first line to the second, and the bound of Chandrasekharan and Narasimhan is used to pass from the second to the third. Taking the square root of each side and dividing by X gives

$$\frac{1}{X}\sum_{n\leq X} |S_f(n)| \ll X^{\frac{k-1}{2}+\frac{1}{4}},$$

which is precisely the statement that the Classical Conjecture holds *on average*. Their result is described more completely in §3.2.4.

Building on this result, Hafner and Ivić were able to show that for holomorphic cusp forms of full integral weight on $\mathrm{SL}(2,\mathbb{Z})$, we know

$$S_f(n) \ll X^{\frac{k-1}{2}+\frac{1}{3}}.$$

The argument of Hafner and Ivić only applies for holomorphic forms of full-integral eight and of level one, but it is possible to provide some extension to their result using their methodology.

In the rest of this chapter, we will examine a new method for examining the behavior of $S_f(n)$. We will be able to study a slightly more general object. Let $g = \sum_{n\geq 1} b(n)e(nz)$ be another modular form of weight k and of the same level as f. The fundamental idea is to study the Dirichlet series

$$D(s, S_f) = \sum_{n\geq 1} \frac{S_f(n)}{n^{s+\frac{k-1}{2}}}$$
$$D(s, S_f \times S_g) = \sum_{n\geq 1} \frac{S_f(n)S_g(n)}{n^{s+k-1}}$$
$$D(s, S_f \times \overline{S_g}) = \sum_{n\geq 1} \frac{S_f(n)\overline{S_g(n)}}{n^{s+k-1}}.$$

In the sequel, we show that these three Dirichlet series have meromorphic continuation to $\mathbb{C}$. In §3.5, we show how to analyze these Dirichlet series to prove results concerning average sizes of the partial sums $S_f(n)$.

Remark 3.1.2. Note that the notation used in this thesis is different than the notation used in the series of papers [HKLDW17a, HKLDW17c, HKLDW16]. In this thesis, we adopt the convention that has risen amidst the representation theoretic point of view on automorphic forms. Therefore $L(s, f \times f)$ in this thesis is the same as $L(s, f \times \overline{f})$ in the papers. This difference is ultimately very minor, and does not change any aspect of the analysis.

3.2 Useful Tools and Notation Reference

For ease of reference, we give a notational reference and a brief description of some of the tools necessary for the analysis.

3.2.1 The Rankin–Selberg L-function

The Rankin–Selberg convolution L-function is described in detail in [Gol06, Bum98], but we summarize its construction and properties. Note that there is a choice of convention concerning notation for conjugation. The more common convention is changing due to influence from more general lines of inquiry.

Let $f(z) = \sum a(n)e(nz)$ and $g(z) = \sum b(n)e(nz)$ be modular forms of weight k on a congruence subgroup $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$, where at least one is cuspidal. Let $\Gamma\backslash\mathcal{H}$ denote the upper half plane modulo the group action due to Γ, and let $\langle f, g\rangle$ denote the Petersson inner product

$$\langle f, g\rangle = \iint_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)}\frac{dxdy}{y^2}.$$

The Rankin–Selberg L-function is given by the Dirichlet series

$$L(s, f\times\overline{g}) := \zeta(2s)\sum_{n\geq 1}\frac{a(n)\overline{b(n)}}{n^{s+k-1}},$$

which is absolutely convergent for $\mathrm{Re}\, s > 1$. This L-function has a meromorphic continuation to all $s\in\mathbb{C}$ via the identity

$$L(s, f\times\overline{g}) := \frac{(4\pi)^{s+k-1}\zeta(2s)}{\Gamma(s+k-1)}\langle \mathrm{Im}(\cdot)^k f\overline{g}, E(\cdot,\overline{s})\rangle, \tag{2.1}$$

where $E(z,s)$ is the real-analytic Eisenstein series

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\mathrm{Im}(\gamma z)^s.$$

If, in (2.1), we replace $f\overline{g}$ with $fT_{-1}g$, where T_{-1} is the Hecke operator giving the action

$$T_{-1}F(x+iy) = F(-x+iy),$$

then one gets a meromorphic continuation of

$$L(s, f\times g) = \zeta(2s)\sum_{n\geq 1}\frac{a(n)b(n)}{n^{s+k-1}}.$$

More details on the T_{-1} Hecke operator can be found in the discussion leading up to Theorem 3.12.6 of [Gol06]. As the meromorphic properties of both are determined by the zeta function, Gamma function, and Eisenstein series, we see that complex analytic arguments are very similar on either $L(s, f \times \overline{g})$ or $L(s, f \times g)$. We will only carry out the argument for $L(s, f \times \overline{g})$, and describe any changes necessary to perform the argument on the other.

These Rankin–Selberg L-functions are holomorphic except for, at most, a simple pole at $s = 1$ with residue proportional to $\langle f, g\rangle$. When $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, there is the functional equation

$$(2\pi)^{-2s}\Gamma(s)\Gamma(s + k - 1)L(s, f \times \overline{g}) =: \Lambda(s, f \times \overline{g}) = \Lambda(1 - s, f \times \overline{g}),$$

coming from the functional equation of the completed Eisenstein series

$$\pi^{-s}\Gamma(s)\zeta(2s)E(z, s) =: E^*(z, s) = E^*(z, 1 - s).$$

There are analogous transformations for higher levels, but their formulation is a bit more complicated due to the existence of other cusps.

3.2.2 Selberg spectral expansion

Let $L^2(\Gamma\backslash\mathcal{H})$ denote the space of square integrable functions on $\Gamma\backslash\mathcal{H}$ with respect to the Petersson norm. There is a complete orthonormal system for the residual and cuspidal spaces of $\Gamma\backslash\mathcal{H}$, which we denote by $\{\mu_j(z) : j \geq 0\}$, consisting of the constant function $\mu_0(z)$ and infinitely many Maass cusp forms $\mu_j(z)$ for $j \geq 1$ with associated eigenvalues $\frac{1}{4} + t_j^2$ with respect to the hyperbolic Laplacian. Without loss of generality, we assume that the μ_j are also simultaneous eigenfunctions of the standard Hecke operators, as well as the T_{-1} operator. Then for any $f \in L^2(\Gamma\backslash\mathcal{H})$, we have the *spectral decomposition* of f given by

$$f(z) = \sum_j \langle f, \mu_j\rangle\mu_j(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{\mathbb{R}} \langle f, E_{\mathfrak{a}}(\cdot, \tfrac{1}{2} + it)\rangle E_{\mathfrak{a}}(a, \tfrac{1}{2} + it)\, dt,$$

where $\mathfrak{a}$ ranges of the cusps of $\Gamma\backslash\mathcal{H}$. Throughout this thesis, we will refer to the first sum as the *discrete spectrum* and the sums of integrals as the *continuous spectrum*. The spectral decomposition as presented here is a consequence of Selberg's Spectral Theorem, as presented in Theorem 15.5 of [IK04].

To each Maass form μ_j is associated a type $\frac{1}{2} + it_j$, and these it_j are expected to satisfy Selberg's Eigenvalue Conjecture, which says that all t_j are real. In complete generality, it is known that t_j is either purely real or purely imaginary. Selberg's Eigenvalue Conjecture has been proved for many congruence subgroups, including

$\mathrm{SL}(2,\mathbb{Z})$, but it is not known in general. We let $\theta = \sup_j\{|\mathrm{Im}(t_j)|\}$ denote the best known progress towards Selberg's Eigenvalue Conjecture for Γ. The current best result for θ in all congruence subgroups that $\theta \leq \frac{7}{64}$, due to Kim and Sarnak [KS03].

3.2.3 Decoupling integral transform

We will use an integral analogue of the binomial theorem, originally considered by Barnes [Bar08], also presented in 6.422(3) of [JZ07].

Lemma 3.2.1 (Barnes, 1908)**.** *If* $0 > \gamma > -\operatorname{Re} s$ *and* $|\arg t| < \pi$*, then*

$$\frac{1}{2\pi i}\int_{(\gamma)} \Gamma(-z)\Gamma(s+z)t^z\, dz = \Gamma(s)(1+t)^{-s}.$$

This is a corollary to an integral representation of the beta function,

$$B(z,s) = \int_0^\infty \frac{x^z}{(1+x)^{z+s}}\,\frac{dx}{x},$$

which implies that

$$B(z,s-z) = \int_0^\infty \frac{x^z}{(1+x)^{s}}\,\frac{dx}{x}. \tag{2.2}$$

The right hand side is a Mellin transform, so (2.2) indicates that $B(z,s-z)$ is the Mellin transform of $(1+x)^{-s}$ (with auxiliary variable z). Applying the Mellin Inversion Theorem (as shown in [Tit86]) and the representation of the Beta function in terms of Gamma functions, $B(s,t) = \Gamma(s)\Gamma(t)/\Gamma(s+t)$, we recover a proof of the Lemma.

We will apply this integral transform to decouple m, n in $(m+n)^{-s} = m^{-s}(1+\frac{n}{m})^{-s}$. It is easy to check that an application of the lemma (followed by a change of variables $z \mapsto -z$) gives

$$\frac{1}{(n+m)^s} = \frac{1}{2\pi i}\int_{(\gamma)} \frac{\Gamma(z)\Gamma(s-z)}{\Gamma(s)}\frac{1}{n^{s-z}m^z}\, dz.$$

3.2.4 Chandrasekharan and Narasimhan

We will refer to a result of Chandrasekharan and Narasimhan through this thesis. We combine [CN62, Theorem 4.1] and [CN64, Theorem 1] to state the following theorem of theirs.

Theorem 3.2.2 (Chandrasekharan and Narasimhan, 1962 and 1964)**.** *Let f and g be objects with meromorphic Dirichlet series*

$$L(s,f)=\sum_{n\geq 1}\frac{a(n)}{n^s},\qquad L(s,g)=\sum_{n\geq 1}\frac{b(n)}{n^s}.$$

Suppose $G(s)=Q^s\prod_{i=1}^{\ell}\Gamma(\alpha_i s+\beta_i)$ is a product of Gamma factors with $Q>0$ and $\alpha_i>0$. Define $A=\sum_{i=1}^{\ell}\alpha_i$. Let w and w' be numbers such that $\sum_{n\leq X}|b(n)|^2\ll X^{2w-1}\log^{w'}X$. Let

$$Q(X)=\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{L(s,f)}{s}X^s\,ds,$$

where $\mathcal{C}$ is a smooth closed contour enclosing all the singularities of the integrand. Let q be the maximum of the real parts of the singularities of $L(s,f)$ and let r be the maximum order of a pole of $L(s,f)$ with real part q. Suppose the functional equation

$$G(s)L(s,f)=\epsilon(f)G(\delta-s)L(\delta-s,g)$$

is satisfied for some $|\epsilon(f)|=1$ and $\delta>0$. Then we have that

$$\begin{aligned}S_f(X)=\sum_{n\leq X}a(n)&=Q(X)+O(X^{\frac{\delta}{2}-\frac{1}{4A}+2A(w-\frac{\delta}{2}-\frac{1}{4A})\eta+\epsilon})\\&\quad+O(X^{q-\frac{1}{2A}-\eta}\log(X)^{r-1})+O\Big(\sum_{X\leq n\leq X'}|a(n)|\Big)\end{aligned}$$

for any $\eta\geq 0$, and where $X'=X+O(X^{1-\frac{1}{2A}-\eta})$. If all $a(n)\geq 0$, the final O-error term above does not contribute.

Suppose further that $A\geq 1$ and that $2w-\delta-\frac{1}{A}\leq 0$. Then

$$\sum_{n\leq X}|S_f(n)-Q(n)|^2=cX^{\delta-\frac{1}{2A}+1}+O(X^{\delta}\log^{w'+2}X)$$

for a constant c that can be made explicit.

Thus from little more than a functional equation with understood Gamma factors, one can produce nontrivial bounds on first and second moments.

3.3 Meromorphic Continuation

We will now produce the meromorphic continuations of $D(s,S_f)$ and $D(s,S_f\times\overline{S_g})$. The meromorphic continuation of $D(s,S_f\times S_g)$ follows from applying the exact same

methodology to $T_{-1}g$ in place of g, so we only write down the corresponding results. In this section, we explicitly show the results in the case when $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. For higher level congruence subgroups, the same methodology will work. We remark on this in §3.6.

Throughout this section, let $f(z) = \sum a(n)e(nz)$ and $g(z) = \sum b(n)e(nz)$ be weight k cusp forms on $\mathrm{SL}(2,\mathbb{Z})$. Define $S_f(n) := \sum_{m\leq n} a(m)$ to be the partial sum of the first n Fourier coefficients of f. Define $S_g(n)$ similarly.

We first describe the meromorphic continuation of $D(s, S_f)$. The main ideas of this continuation are very similar to those in the continuation of $D(s, S_f \times \overline{S_g})$, but the details are much simpler. We then produce the meromorphic continuation of $D(s, S_f \times \overline{S_g})$ in §3.3.2. We shall see that the main obstacle is understanding the shifted convolution sum

$$\sum_{n,h\geq 1} \frac{a(n)\overline{b(n-h)}}{n^{s+k-1}h^w}.$$

We then use these meromorphic continuations to understand $D(s, S_f \times \overline{S_g})$ in the next section.

3.3.1 Meromorphic continuation of $D(s, S_f)$

The meromorphic continuation of $D(s, S_f)$ is very simple. The pattern of the proof is similar to the pattern necessary for $D(s, S_f \times \overline{S_g})$, so it is useful to be very clear. The proof proceeds in two steps:

(i) Decompose the Dirichlet series into sums of Dirichlet series that are easier to understand, and

(ii) Understand the reduced Dirichlet series by relating them to L-functions.

Proposition 3.3.1. *With f and $S_f(n)$ as defined above, the Dirichlet series associated to $S_f(n)$ decomposes into*

$$D(s, S_f) := \sum_{n\geq 1} \frac{S_f(n)}{n^{s+\frac{k-1}{2}}} = L(s,f) + \frac{1}{2\pi i}\int_{(2)} L(s-z,f)\zeta(z)\frac{\Gamma(z)\Gamma(s+\frac{k-1}{2}-z)}{\Gamma(s+\frac{k-1}{2})}\,dz, \tag{3.1}$$

valid for $\mathrm{Re}\, s > 3$. Here, $L(s,f)$ denotes the standard L-function associated to f, given by

$$L(s,f) := \sum_{n\geq 1} \frac{a(n)}{n^{s+\frac{k-1}{2}}},$$

normalized to have functional equation of the form $s \mapsto 1-s$.

Proof. We directly manipulate the Dirichlet series.

$$D(s, S_f) = \sum_{n\geq 1} \frac{S_f(n)}{n^{s+\frac{k-1}{2}}} = \sum_{\substack{n\geq 1\\ m\geq 0}} \frac{a(n-m)}{n^{s+\frac{k-1}{2}}}.$$

In this last equality, we adopt the convention that $a(n) = 0$ for $n \leq 0$ to simplify notation. Separate the $m = 0$ case and reindex the remaining sum with $n \mapsto n + m$ to get

$$\sum_{n\geq 1} \frac{a(n)}{n^{s+\frac{k-1}{2}}} + \sum_{m,n\geq 1} \frac{a(n)}{(n+m)^{s+\frac{k-1}{2}}}.$$

The first sum is exactly $L(s, f)$. In the second sum, decouple $(n+m)^{-s}$ through the use of the Mellin-Barnes transform detailed in §3.2.3. For $\gamma > 1$ and $\operatorname{Re} s$ sufficiently large, the m sum can be collected into $\zeta(z)$ and the n sum can be collected into $L(s + \frac{k-1}{2})$. Simplification completes the proof. □

The meromorphic continuation of the L-function $L(s, f)$ is well-understood. Note that in

$$\frac{1}{2\pi i} \int_{(2)} L(s-z, f)\zeta(z) \frac{\Gamma(z)\Gamma(s+\frac{k-1}{2}-z)}{\Gamma(s+\frac{k-1}{2})}\, dz,$$

the integrand is meromorphic in both s and z, and has exponential decay in vertical strips in $\operatorname{Im} z$ for any individual s. Therefore one can use the meromorphic continuation of $L(s, f)$ to understand that this integral is meromorphic for all $s \in \mathbb{C}$.

Note that it is also possible to shift the line of z integration arbitrarily far in the negative direction, passing poles at $z = 1, 0, -1, \ldots$ and picking up their residues. Shifting the line of z integration to ϵ for a small $\epsilon > 0$ passes exactly one pole, coming from $\zeta(z)$ at $z = 1$, with residue

$$\operatorname*{Res}_{z=1} = L(s-1, f) \frac{\Gamma(s+\frac{k-1}{2}-1)}{\Gamma(s+\frac{k-1}{2})} = \frac{L(s-1, f)}{s+\frac{k-1}{2}-1}.$$

Therefore, we have the equality

$$D(s, S_f) = L(s, f) + \frac{L(s-1, f)}{s+\frac{k-1}{2}-1} + \frac{1}{2\pi i} \int_{(\epsilon)} L(s-z, f)\zeta(z) \frac{\Gamma(z)\Gamma(s+\frac{k-1}{2}-z)}{\Gamma(s+\frac{k-1}{2})}\, dz.$$

The first term is analytic, the third term is analytic in s for $\operatorname{Re} s > \epsilon - \frac{k-1}{2}$, and the middle term appears to have a simple pole at $s = 1 - \frac{k-1}{2}$. However, from the functional equation equation of $L(s, f)$,

$$\Lambda(s, f) := (2\pi)^{-(s+\frac{k-1}{2})}\Gamma(s+\tfrac{k-1}{2})L(s, f) = \varepsilon\Lambda(1-s, f),$$

we see that the residue $L(-\frac{k-1}{2})$ is a trivial zero of the L-function, and $s = 1 - \frac{k-1}{2}$ is not a pole after all.

Further shifting the line of z integration to $-M + \epsilon$ for small $\epsilon > 0$ passes (M) further poles, coming from $\Gamma(z)$ at $z = -j$ for $0 \leq j < M$. The jth pole, located at $z = -j, 0 \leq j < M$ has residue

$$\operatorname*{Res}_{z=-j} = L(s+j, f)\zeta(-j)\frac{\Gamma(s + \frac{k-1}{2} + j)}{\Gamma(s + \frac{k-1}{2})}\frac{(-1)^j}{j!}.$$

In each residue, the L-function is analytic, and all apparent poles of the Gamma function in the numerator are cancelled by poles of the Gamma function in the denominator. Therefore each residue is analytic in s.

For any integer $M \geq 0$, we therefore have that

$$D(s, S_f) = L(s, f) + \sum_{j=-1}^{M-1} \operatorname*{Res}_j + \frac{1}{2\pi i}\int_{(-M+\epsilon)} L(s-z, f)\zeta(z)\frac{\Gamma(z)\Gamma(s + \frac{k-1}{2} - z)}{\Gamma(s + \frac{k-1}{2})}dz.$$

The L-function and residues are analytic in s. The integral term is analytic in s for $\operatorname{Re} s > -\frac{k-1}{2} - M + \epsilon$. Since M is arbitrary, we see that $D(s, S_f)$ is actually analytic for all $s \in \mathbb{C}$. We record this observation as a corollary to the decomposition of $D(s, S_f)$.

Corollary 3.3.2. *The Dirichlet series $D(s, S_f)$ has analytic continuation to the whole complex plane, given by (3.1).*

Remark 3.3.3. I note that if f is not cuspidal, then the decomposition and most of the analysis of $D(s, S_f)$ carries over verbatim, with one key difference: the value $L(-\frac{k-1}{2})$ is no longer a trivial zero. Therefore it is possible to show in general that $D(s, S_f)$ is meromorphic in the plane with at most several simple poles with residues given by special values of $L(s, f)$.

This indicates a very strong parallel between the properties of $L(s, f)$ and $D(s, S_f)$. But it should be noted that $D(s, S_f)$ does not have a simple functional equation or an Euler product.

3.3.2 Meromorphic continuation of $D(s, S_f \times \overline{S_g})$

The meromorphic continuation of $D(s, S_f \times \overline{S_g})$ is a bit involved, but the approach is very similar to the approach for $D(s, S_f)$. We proceed in three steps:

(i) Decompose $D(s, S_f \times \overline{S_g})$ into sums of Dirichlet series that are easier to understand,

(ii) Related the reduced Dirichlet series to L-functions and convolution sums, and

(iii) Combine the analytic properties of the L-functions and convolution sums.

The first two steps are fundamentally the same as in$D(s, S_f)$, except that convolution sums are necessary in the analysis. As the meromorphic properties of convolution sums are significantly more delicate, ascertaining the final analytic properties will take much more work. The final step is deferred to §3.4.

Proposition 3.3.4. *With $f, g, S_f(n)$, and $S_g(n)$ as defined above, the Dirichlet series associated to $S_f(n)\overline{S_g(n)}$ decomposes into*

$$\begin{aligned} D(s,S_f \times \overline{S_g}) &:= \sum_{n\geq 1} \frac{S_f(n)\overline{S_g(n)}}{n^{s+k-1}} \\ &= W(s; f, \overline{g}) + \frac{1}{2\pi i}\int_{(\gamma)} W(s-z; f, \overline{g})\zeta(z)\frac{\Gamma(z)\Gamma(s-z+k-1)}{\Gamma(s+k-1)}\, dz, \end{aligned}$$

for $1 < \gamma < \mathrm{Re}(s-1)$. Here, $W(s; f, \overline{g})$ denotes

$$W(s; f, \overline{g}) := \frac{L(s, f \times \overline{g})}{\zeta(2s)} + Z(s, 0, f \times \overline{g}),$$

$L(s, f \times \overline{g})$ denotes the Rankin–Selberg L-function as in 3.2.1, and $Z(s, w, f \times \overline{g})$ denotes the symmetrized shifted convolution sum

$$Z(s, w, f \times \overline{g}) := \sum_{n,h\geq 1} \frac{a(n)\overline{b(n-h)} + a(n-h)\overline{b(n)}}{n^{s+k-1}h^w}.$$

Proof. Expand and recollect the partial sums S_f and S_g.

$$D(s, S_f \times \overline{S_g}) = \sum_{n\geq 1} \frac{S_f(n)\overline{S_g(n)}}{n^{s+k-1}} = \sum_{n\geq 1} \frac{1}{n^{s+k-1}} \sum_{m=1}^{n} a(m) \sum_{h=1}^{n} \overline{b(h)}.$$

Separate the sums over m and h into the cases where $m = h, m > h$, and $m < h$. We again adopt the convention that $a(n) = 0$ for $n \leq 0$ to simplify notation. Reorder the sums, summing down from n instead of up to n, giving

$$\sum_{n\geq 1} \frac{1}{n^{s+k-1}} \left(\sum_{h=m>0} + \sum_{h>m\geq 0} + \sum_{m>h\geq 0} \right) a(n-m)\overline{b(n-h)}.$$

In the first sum, take $h = m$. In the second sum, when $h > m$, we let $h = m + \ell$ and then sum over m and ℓ. Similarly in the third sum, when $m > h$, we let $m = h + \ell$.

Together, this yields

$$\sum_{n\geq 1}\frac{1}{n^{s+k-1}}\Bigg(\sum_{m\geq 0}a(n-m)\overline{b(n-m)} + \sum_{\substack{\ell\geq 1\\ m\geq 0}}a(n-m)\overline{b(n-m-\ell)} + \sum_{\substack{\ell\geq 1\\ m\geq 0}}a(n-m-\ell)\overline{b(n-m)}\Bigg).$$

In each sum, the cases when $m = 0$ are distinguished. Altogether, these contribute

$$W(s; f, \overline{g}) = \frac{L(s, f\times\overline{g})}{\zeta(2s)} + Z(s, 0, f\times\overline{g}).$$

Within $W(s; f, \overline{g})$, one should think of $L(s, f\times\overline{g})\zeta(2s)^{-1}$ as the diagonal part of the double summation, while $Z(s, 0, f\times\overline{g})$ contains the off-diagonal, written as the sum of the above-diagonal and below-diagonal parts of the double summation.

Reindexing by changing $n \mapsto n + m$, the remaining sums with $m \geq 1$ can be rewritten as

$$\sum_{m,n\geq 1}\frac{1}{(n+m)^{s+k-1}}\Bigg(a(n)\overline{b(n)} + \sum_{\ell\geq 1}a(n)\overline{b(n-\ell)} + \sum_{\ell\geq 1}a(n-\ell)\overline{b(n)}\Bigg).$$

Decouple $(n + m)^{-(s+k-1)}$ through the use of the Mellin-Barnes transform detailed in §3.2.3. Restricting to $\gamma > 1$ and $\operatorname{Re} s$ sufficiently large, we can collect the m sum into $\zeta(z)$ and the n sum can be colleced into $W(s; f, g)$. Simplification completes the proof. □

As in the case of $D(s, S_f)$, the individual pieces $L(s, f\times\overline{g})\zeta(2s)^{-1}$ and $Z(s, w, f\times\overline{g})$ are known to have meromorphic continuation to the complex plane. The Rankin–Selberg L-function is classical, and its meromorphic continuation is explained in §3.2.1. The meromorphic properties of $Z(s, w, f\times\overline{g})$ are extensively studied in [HHR13]. One should expect to be able to perform an analysis similar to the analysis for $D(s, S_f)$ to study $D(s, S_f\times\overline{S_g})$, perhaps by shifting the line of z integration and analyzing residue terms.

However, the shifted sums $Z(s, 0, f\times\overline{g})$ show miraculous cancellation with the diagonal $L(s, f\times\overline{g})\zeta(2s)^{-1}$ that does not occur in $Z(s, w, f\times\overline{g})$ for general w. We group $W(s; f, \overline{g})$ into a single object and study its analytic properties in the next section.

3.4 Analytic Behavior of $D(s, S_f \times \overline{S_g})$

In this section, we will understand the meromorphic continuation of $D(s, S_f \times \overline{S_g})$ by studying the analytic properties of $W(s; f, \overline{g})$. Although we work in level 1, the methodology in this section generalizes to arbitrary level and half-integral weight forms. Therefore, we will use $\theta = \sup_j\{|\mathrm{Im}(t_j)|\}$ to denote progress towards Selberg's Eigenvalue Conjecture (as described in §3.2.2) even though it is known that $\theta = 0$ in the level 1 case.

We first produce a spectral expansion for the off-diagonal component, the symmetrized shifted double Dirichlet series

$$Z(s, w, f \times \overline{g}) := \sum_{m\geq 1}\sum_{\ell\geq 1} \frac{a(m)\overline{b(m-\ell)} + a(m-\ell)\overline{b(m)}}{m^{s+k-1}\ell^w}.$$

We then use this to understand the analytic behavior of $W(s; f, \overline{g})$ and, from this, the analytic behavior of $D(s, S_f \times \overline{S_g})$.

Spectral expansion

For each integer $h \geq 1$, define the weight zero Poincaré series on Γ,

$$P_h(z, s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s e\left(h\gamma z\right),$$

defined initially for $\mathrm{Re}(s)$ sufficiently large, but with meromorphic continuation to all $s \in \mathbb{C}$.

Recall T_{-1} from §3.2.1. Let $\mathcal{V}_{f,\overline{g}}(z) := y^k(f\overline{g} + T_{-1}(f\overline{g}))$. Note that $\mathcal{V}_{f,\overline{g}}(z) \in L^2(\Gamma\backslash\mathcal{H})$, so the Petersson inner product $\langle \mathcal{V}_{f,\overline{g}}, P_h(\cdot, \overline{s})\rangle$ converges. By expanding this inner product, we get

$$\langle \mathcal{V}_{f,\overline{g}}, P_h(\cdot, \overline{s})\rangle = \frac{\Gamma\left(s+k-1\right)}{(4\pi)^{s+k-1}} D_{f,\overline{g}}(s; h),$$

where we define $D_{f,\overline{g}}$ to have analogous notation as in [HHR13],

$$D_{f,g}(s; h) := \sum_{n\geq 1} \frac{a(n)\overline{b(n-h)} + a(n-h)\overline{b(n)}}{n^{s+k-1}},$$

which converges absolutely for $\mathrm{Re}\, s$ sufficiently positive. Dividing by h^w and summing over $h \geq 1$ recovers $Z(s, w, f \times \overline{g})$,

$$Z(s, w, f \times \overline{g}) := \sum_{n,h\geq 1} \frac{D_{f,\overline{g}}(s; h)}{h^w} = \frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)} \sum_{h\geq 1} \frac{\langle \mathcal{V}_{f,\overline{g}}, P_h\rangle}{h^w}, \tag{4.1}$$

for $\operatorname{Re} s$ and $\operatorname{Re} w$ sufficiently positive.

We will obtain a meromorphic continuation of $Z(s, w, f \times \overline{g})$ by using the spectral expansion of the Poincaré series and substituting it into (4.1). Let $\{\mu_j\}$ be an orthonormal basis of Maass eigenforms with associated types $\frac{1}{2} + it_j$ for $L^2(\Gamma\backslash\mathcal{H})$ as in §3.2.2, each with Fourier expansion

$$\mu_j(z) = \sum_{n \neq 0} \rho_j(n) y^{\frac{1}{2}} K_{it_j}(2\pi|n|y) e^{2\pi i n x}.$$

The spectral expansion of the Poincaré series is given by

$$\begin{aligned} P_h(z, s) = &\sum_j \langle P_h(\cdot, s), \mu_j \rangle \mu_j(z) \\ &+ \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle P_h(\cdot, s), E(\cdot, \tfrac{1}{2} + it) \rangle E(z, \tfrac{1}{2} + it)\, dt. \end{aligned} \tag{4.2}$$

We shall refer to the above sum and integral as the discrete and continuous spectrum, respectively, similar to the convention in §3.2.2.

The inner product of μ_j against the Poincaré series gives

$$\langle P_h(\cdot, s), \mu_j \rangle = \frac{\overline{\rho_j(h)}\sqrt{\pi}}{(4\pi h)^{s-\frac{1}{2}}} \frac{\Gamma(s - \frac{1}{2} + it_j)\Gamma(s - \frac{1}{2} - it_j)}{\Gamma(s)}. \tag{4.3}$$

Remark 3.4.1. In the computation of this inner product and the inner product of the Eisenstein series against the Poincaré series, we use formula [JZ07, §6.621(3)] to evaluate the final integrals.

Let $E(z, w)$ be the Eisenstein series on $\mathrm{SL}(2, \mathbb{Z})$, given by

$$E(z, w) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} (\operatorname{Im} \gamma z)^s.$$

Then $E(z, w)$ has Fourier expansion (as in [Gol06, Chapter 3])

$$\begin{aligned} E(z, w) = &y^w + \phi(w) y^{1-w} \\ &+ \frac{2\pi^w \sqrt{y}}{\Gamma(w)\zeta(2w)} \sum_{m \neq 0} |m|^{w-\frac{1}{2}} \sigma_{1-2w}(|m|) K_{w-\frac{1}{2}}(2\pi|m|y) e^{2\pi i m x}, \end{aligned} \tag{4.4}$$

where

$$\phi(w) = \sqrt{\pi} \frac{\Gamma(w - \frac{1}{2})\zeta(2w - 1)}{\Gamma(w)\zeta(2w)}.$$

The Petersson inner product of the Poincaré series ($h \geq 1$) against the Eisenstein series $E(z, w)$ is given by

$$\langle P_h(\cdot, s), E(\cdot, w)\rangle = \frac{2\pi^{\overline{w}+\frac{1}{2}} h^{\overline{w}-\frac{1}{2}} \sigma_{1-2\overline{w}}(h)}{\zeta(2\overline{w})(4\pi h)^{s-\frac{1}{2}}} \frac{\Gamma(s+\overline{w}-1)\Gamma(s-\overline{w})}{\Gamma(\overline{w})\Gamma(s)}, \tag{4.5}$$

provided that $\operatorname{Re} s > \frac{1}{2} + |\operatorname{Re} w - \frac{1}{2}|$. For t real, $w = \frac{1}{2} + it$, and $\operatorname{Re} s > \frac{1}{2}$, we can specialize (4.5) to

$$\langle P_h(\cdot, s), E(\cdot, \tfrac{1}{2} + it)\rangle = \frac{2\sqrt{\pi}\sigma_{2it}(h)}{\Gamma(s)(4\pi h)^{s-\frac{1}{2}}} \frac{\Gamma(s-\frac{1}{2}+it)\Gamma(s-\frac{1}{2}-it)}{h^{it}\zeta^*(1-2it)}, \tag{4.6}$$

in which $\zeta^*(2s) := \pi^{-s}\Gamma(s)\zeta(2s)$ denotes the completed zeta function.

Now that we have computed the inner products of the Eisenstein series and Maass forms with the Poincaré series, we are ready to analyze the spectral expansion. After substituting (4.3) into the discrete part of (4.2), the discrete spectrum takes the form

$$\frac{\sqrt{\pi}}{(4\pi h)^{s-\frac{1}{2}}\Gamma(s)} \sum_j \overline{\rho_j(h)}\Gamma(s-\tfrac{1}{2}+it_j)\Gamma(s-\tfrac{1}{2}-it_j)\mu_j(z)$$

and is analytic in s in the right half-plane $\operatorname{Re} s > \frac{1}{2} + \theta$. After inserting (4.6), the continuous spectrum takes the form

$$\frac{\sqrt{\pi}}{2\pi(4\pi h)^{s-\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\sigma_{2it}(h)}{h^{it}} \frac{\Gamma(s-\frac{1}{2}+it)\Gamma(s-\frac{1}{2}-it)}{\zeta^*(1-2it)\Gamma(s)} E(z, \tfrac{1}{2}+it)\, dt,$$

which is analytic in s for $\operatorname{Re} s > \frac{1}{2}$ and has apparent poles when $\operatorname{Re} s = \frac{1}{2}$.

Substiting this spectral expansion into (4.1) and summing over $h \geq 1$ recovers an expression for all of $Z(s, w, f \times \overline{g})$. Recognizing the Dirichlet series (as described in [Gol06], for instance)

$$\sum_{h\geq 1} \frac{\rho_j(h)}{h^{s+w-\frac{1}{2}}} = L(s+w-\tfrac{1}{2}, \mu_j)$$
$$\sum \frac{\sigma_{1-2w}(h)}{h^{s+\frac{1}{2}-w}} = L(s, E(\cdot, w)) = \zeta(s+w-\tfrac{1}{2})\zeta(s-w+\tfrac{1}{2}),$$

we are able to execute the h sum completely. Assembling it all together, we have proved the following proposition.

Proposition 3.4.2. *For f, g weight k forms on $\mathrm{SL}_2(\mathbb{Z})$, the shifted convolution sum $Z(s, w, f \times \overline{g})$ can be expressed as*

$$Z(s, w, f \times \overline{g}) := \sum_{m=1}^{\infty} \frac{a(m)\overline{b(m-h)} + a(m-h)\overline{b(m)}}{m^{s+k-1}h^w}$$

$$= \frac{(4\pi)^k}{2} \sum_j \rho_j(1) G(s, it_j) L(s + w - \tfrac{1}{2}, \mu_j) \langle \mathcal{V}_{f,\overline{g}}, \mu_j \rangle \tag{4.7}$$

$$+ \frac{(4\pi)^k}{4\pi i} \int_{(0)} G(s, z) \mathcal{Z}(s, w, z) \langle \mathcal{V}_{f,\overline{g}}, E(\cdot, \tfrac{1}{2} - \overline{z}) \rangle \, dz, \tag{4.8}$$

when $\mathrm{Re}(s + w) > \frac{3}{2}$, *where* $G(s, z)$ *and* $\mathcal{Z}(s, w, z)$ *are the collected* Γ *and* ζ *factors of the discrete and continuous spectra,*

$$G(s, z) = \frac{\Gamma(s - \frac{1}{2} + z)\Gamma(s - \frac{1}{2} - z)}{\Gamma(s)\Gamma(s + k - 1)}$$

$$\mathcal{Z}(s, w, z) = \frac{\zeta(s + w - \frac{1}{2} + z)\zeta(s + w - \frac{1}{2} - z)}{\zeta^*(1 + 2z)}.$$

Remark 3.4.3. Let's verify that this spectral expansion converges. Recall Stirling's approximation: for $x, y \in \mathbb{R}$,

$$\gamma(x + iy) \sim (1 + |y|)^{x - \frac{1}{2}} e^{-\frac{\pi}{2}|y|}$$

as $y \to \pm\infty$ with x bounded. For vertical strips in s and z,

$$G(s, z) \sim P(s, z) e^{-\frac{\pi}{2}(2\max(|s|,|z|) - 2|s|)},$$

where $P(s, z)$ has at most polynomial growth in s and z. When k is a full-integer, Watson's triple product formula (given in Theorem 3 of [Wat08]) shows that

$$\rho_j(1) \langle f\overline{g} \, \mathrm{Im}(\cdot)^k, \overline{\mu_j} \rangle, \quad \text{and} \quad \rho_j(1) \langle T_{-1}(f\overline{g}) \, \mathrm{Im}(\cdot)^k, \overline{\mu_j} \rangle$$

has at most polynomial growth in $|t_j|$. When k is a half-integer, Kıral's bound (given in Proposition 13 of [K15]) also proves polynomial growth, albeit of a higher degree. Through direct computation with the associated Rankin–Selberg L-function, the same can be said about

$$\langle \mathcal{V}_{f,\overline{g}}, E(\cdot, \tfrac{1}{2} + z) \rangle / \zeta^*(1 + 2z).$$

Both (4.7) and (4.8) converge uniformly on vertical strips in t_j and have at most polynomial growth in s.

We will now specialize to $w = 0$ and analyze the meromorphic properties of $Z(s, 0, f \times \overline{g})$. This very naturally breaks into two parts: the contribution from the discrete spectrum (in line (4.7)) and the contribution from the continuous spectrum (in line (4.8)).

Meromorphic continuation of $Z(s, 0, f \times \overline{g})$: discrete spectrum

Examination of line (4.7), the contribution from the discrete spectrum, reveals that the poles come only from $G(s, it_j)$. There are apparent poles when $s = \frac{1}{2} \pm it_j - n$ for

$n \in \mathbb{Z}_{\geq 0}$. Interestingly, the first set of apparent poles at $s = \frac{1}{2} \pm it_j$ do not actually occur.

Lemma 3.4.4. *For even Maass forms μ_j, we have $L(-2n \pm it_j, \mu_j) = 0$ for $n \in \mathbb{Z}_{\geq 0}$.*

Proof. The completed L-function associated to a Maass form μ_j is given by

$$\Lambda_j(s) = \pi^{-s}\Gamma\left(\tfrac{s+\epsilon+it_j}{2}\right)\Gamma\left(\tfrac{s+\epsilon-it_j}{2}\right)L(s,\mu_j) = (-1)^{\epsilon}\Lambda_j(1-s), \tag{4.9}$$

as in [Gol06, Sec 3.13], where $\epsilon = 0$ if the Maass form μ_j is even and 1 if it is odd. In the case of an even Maass form, the functional equation is of shape

$$\Lambda_j(s) = \pi^{-s}\Gamma\left(\tfrac{s+it_j}{2}\right)\Gamma\left(\tfrac{s-it_j}{2}\right)L(s,\mu_j) = \Lambda_j(1-s).$$

The completed L-function is entire. Thus $L(-2n \pm it_j, \mu_j)$ must be trivial zeroes to cancel the apparent poles at $s = -2n \pm it_j$ from the Gamma functions. □

It turns out that there are no poles appearing from odd Maass forms due to the symmetry of the above-diagonal and below-diagonal terms in $\mathcal{V}_{f,\overline{g}}$.

Lemma 3.4.5. *Suppose f and g are weight k cusp forms, as above. For odd Maass forms μ_j, we have $\langle \mathcal{V}_{f,\overline{g}}, \mu_j \rangle = 0$.*

Proof. Recall that $\mathcal{V}_{f,g} := y^k(f\overline{g} + T_{-1}(f\overline{g}))$, so clearly $T_{-1}\mathcal{V}_{f,\overline{g}} = \mathcal{V}_{f,\overline{g}}$. Recall also that $T_{-1}\mu_j = -\mu_j$ for odd Maass forms μ_j (in fact, this is the defining characteristic of an odd Maass form). Since T_{-1} is a self-adjoint operator with respect to the Petersson inner product we have that

$$\langle \mathcal{V}_{f,\overline{g}}, \mu_j \rangle = \langle T_{-1}\mathcal{V}_{f,\overline{g}}, \mu_j \rangle = \langle \mathcal{V}_{f,\overline{g}}, T_{-1}\mu_j \rangle = -\langle \mathcal{V}_{f,\overline{g}}, \mu_j \rangle.$$

Thus $\langle \mathcal{V}_{f,\overline{g}}, \mu_j \rangle = 0$. □

In the special case when $f = g$, it is possible to show that odd Maass forms μ_j do not contribute poles in *either* the above-diagonal or below-diagonal terms in $\mathcal{V}_{f,\overline{f}}$. We record this observation as a corollary to the above lines of thought, even though it is not necessary for the applications in this thesis.

Corollary. *Suppose f is a full-integral weight cuspidal Hecke eigenform, not necessarily with real coefficients. Then for odd Maass forms μ_j, we have $\langle |f|^2 \operatorname{Im}(\cdot)^k, \mu_j \rangle = 0$. Similarly, we have $\langle f^2 \operatorname{Im}(\cdot)^k, \mu_j \rangle = 0$.*

Proof. We sketch the proof. From Watson's well-known triple product formula [Wat08], we have

$$\langle |f|^2 \operatorname{Im}(\cdot)^k, \mu_j \rangle^2 \sim \frac{L(\frac{1}{2}, f \times \overline{f} \times \mu_j)}{L(1, f, \mathrm{Ad}) L(1, \overline{f}, \mathrm{Ad}) L(1, \mu_j, \mathrm{Ad})}$$

up to multiplication by a nonzero constant coming from the missing Gamma factors. The L-functions in the denominator are all nonzero, and the numerator factors as

$$L(\tfrac{1}{2}, f \times \overline{f} \times \mu_j) = L(\tfrac{1}{2}, \mathrm{Ad}^2 f \times \mu_j) L(\tfrac{1}{2}, \mu_j).$$

Since μ_j is odd, $L(\frac{1}{2}, \mu_j) = 0$ by the functional equation for odd Maass forms, given in (4.9).

Applying Watson's triple product to $\langle f^2 \operatorname{Im}(\cdot)^k, \mu_j \rangle$ yields the numerator

$$L(\tfrac{1}{2}, \mathrm{Sym}^2 f \times \mu_j) L(\tfrac{1}{2}, \mu_j),$$

which is zero for the same reason. □

Lemma 3.4.5 guarantees that the only Maass forms appearing in line (4.7) are even. The first set of apparent poles from even Maass forms appear at $s = \frac{1}{2} \pm i t_j$ and occur as simple poles of the Gamma functions in the numerator of $G(s, t_j)$. They come multiplied by the value of $L(i t_j, \mu_j)$, which by Lemma 3.4.4 is zero.

In summary, $D(s, S_f \times S_g)$ has no poles at $s = \frac{1}{2} \pm i t_j$. The next set of apparent poles are at $s = -\frac{1}{2} \pm i t_j$, appearing at the next set of simple poles of the Gamma functions in the numerator. Unlike the previous poles, these do not coincide with trivial zeroes of the L-function. We have poles of the discrete spectrum at $s = -\frac{1}{2} \pm i t_j$.

Meromorphic continuation of $Z(s, 0, f \times \overline{g})$: continuous spectrum

Examination of the line (4.8), the contribution from the continuous spectrum, is substantially more involved than the discrete spectrum. It is here where the most remarkable cancellation occurs. For ease of reference, we repeat this line, the part we call the continuous spectrum:

$$\frac{(4\pi)^k}{4\pi i} \int_{(0)} G(s, z) \mathcal{Z}(s, w, z) \langle \mathcal{V}_{f, \overline{g}}, E(\cdot, \tfrac{1}{2} - \overline{z}) \rangle \, dz \tag{4.8}$$

where $G(s, z)$ and $\mathcal{Z}(s, w, z)$ are the collected Γ and ζ factors

$$G(s, z) = \frac{\Gamma(s - \frac{1}{2} + z) \Gamma(s - \frac{1}{2} - z)}{\Gamma(s) \Gamma(s + k - 1)}, \quad \mathcal{Z}(s, w, z) = \frac{\zeta(s + w - \frac{1}{2} + z) \zeta(s + w - \frac{1}{2} - z)}{\zeta^*(1 + 2z)}.$$

The rightmost pole seems to occur from the pair of zeta functions in the numerator, occurring when $s+w-\frac{1}{2}\pm z=1$. We must disentangle s and w from z in order to understand these poles.

Line (4.8) is analytic for $\mathrm{Re}(s+w)>\frac{3}{2}$, $\mathrm{Re}\, s>\frac{1}{2}$. As we will shortly set $w=0$, we treat the boundary $\mathrm{Re}(s+w)>\frac{3}{2}$. For s with $\mathrm{Re}\, s\in(\frac{3}{2}-\mathrm{Re}\, w,\frac{3}{2}-\mathrm{Re}\, w+\epsilon)$ for some very small ϵ, we want to shift the contour of integration, avoiding poles coming from the $\zeta^*(1-2z)$ appearing in the denominator of the Fourier expansion of $E(\cdot,\frac{1}{2}+\overline{z})$ (described in (4.4)).

We shift the z-contour to the right while staying within the zero-free region of $\zeta(1-2z)$. By an abuse of notation, we denote this shift here by $\mathrm{Re}\, z=\epsilon$ and let ϵ in this context actually refer to the real value of the z-contour at the relevant imaginary value. This argument can be made completely rigorous, cf. [HHR13, p. 481-483].

We perform this shift in order to guarantee that the two poles in z coming from $\zeta(s+w-\frac{1}{2}\pm z)$, occurring at $\pm z=\frac{3}{2}-s-w$, have different real parts; simultaneously, we pass the pole with more positive real part, occurring at $z=s+w-\frac{3}{2}$ from $\zeta(s+w-\frac{1}{2}-z)$. By the residue theorem,

$$\begin{aligned}&\frac{(4\pi)^k}{4\pi i}\int_{(0)}G(s,z)\mathcal{Z}(s,w,z)\langle\mathcal{V}_{f,\overline{g}},E\rangle\,dz\\&=\frac{(4\pi)^k}{4\pi i}\int_{(\epsilon)}G\mathcal{Z}\langle\mathcal{V}_{f,\overline{g}},E\rangle\,dz-\frac{(4\pi)^k}{2}\operatorname*{Res}_{z=s+w-\frac{3}{2}}G\mathcal{Z}\langle\mathcal{V}_{f,\overline{g}},E\rangle,\end{aligned}$$

where the above residue is found to be

$$-\frac{\zeta(2s+2w-2)\Gamma(2s+w-2)\Gamma(1-w)}{\zeta^*(2s+2w-2)\Gamma(s)\Gamma(s+k-1)}\langle\mathcal{V}_{f,\overline{g}},E(\cdot,2-\overline{s}-\overline{w})\rangle.$$

The residue is analytic in s for $\mathrm{Re}\, s\in(1-\mathrm{Re}\, w,\frac{3}{2}-\mathrm{Re}\, w+\epsilon)$, and has an easily understood meromorphic continuation to the whole plane. Notice also that the shifted contour integral has no poles in s for $\mathrm{Re}\, s\in(\frac{3}{2}-\mathrm{Re}\, w-\epsilon,\frac{3}{2}-\mathrm{Re}\, w+\epsilon)$, so we have found an analytic (not meromorphic!) continuation in s of Line (4.8) past the first apparent pole along $\mathrm{Re}\, s=\frac{3}{2}-\mathrm{Re}\, w$.

For s with $\mathrm{Re}\, s\in(\frac{3}{2}-\mathrm{Re}\, w-\epsilon,\frac{3}{2}-\mathrm{Re}\, w)$, we shift the contour of integration back to $\mathrm{Re}\, z=0$. Since this passes a pole, we pick up a residue. But notice that this is the residue at the *other* pole, $\frac{3}{2}-s-w$, coming from $\zeta(s+w-\frac{1}{2}+z)$,

$$\begin{aligned}&\frac{(4\pi)^k}{4\pi i}\int_{(\epsilon)}G(s,w,z)\mathcal{Z}(s,w,z)\langle\mathcal{V}_{f,\overline{g}},E\rangle dz\\=&\frac{(4\pi)^k}{4\pi i}\int_{(0)}G\mathcal{Z}\langle\mathcal{V}_{f,\overline{g}},E\rangle dz+\frac{(4\pi)^k}{2}\operatorname*{Res}_{z=\frac{3}{2}-s-w}G\mathcal{Z}\langle\mathcal{V}_{f,\overline{g}},E\rangle.\end{aligned}$$

By using the functional equations of the Eisenstein series and zeta functions, one can check that

$$\operatorname*{Res}_{z=\frac{3}{2}-s-w} G\mathcal{Z}\langle \mathcal{V}_{f,\overline{g}}, E\rangle = -\operatorname*{Res}_{z=s+w-\frac{3}{2}} G\mathcal{Z}\langle \mathcal{V}_{f,\overline{g}}, E\rangle,$$

so the two residues combine together, and have well-understood meromorphic continuations to the whole plane. The shifted integral has clear meromorphic continuation until the next apparent poles at $\operatorname{Re} s = \frac{1}{2}$ coming from the Gamma functions $\Gamma(s-\frac{1}{2}\pm z)$ in the integrand. Thus (4.8), originally defined for $\operatorname{Re} s > \frac{3}{2} - \operatorname{Re} w$, has meromorphic continuation for $\frac{1}{2} < \operatorname{Re} s < \frac{3}{2} - \operatorname{Re} w$ given by

$$\frac{(4\pi)^k}{4\pi i}\int_{(0)} G\mathcal{Z}\langle \mathcal{V}_{f,\overline{g}}, E\rangle dz + (4\pi)^k \operatorname*{Res}_{z=\frac{3}{2}-s-w} G\mathcal{Z}\langle \mathcal{V}_{f,\overline{g}}, E\rangle.$$

A very similar argument works to extend the meromorphic continuation in s of the contour integral past the next apparent poles at $\operatorname{Re} s = \frac{1}{2}$ from the Gamma functions, leading to a meromorphic continuation in the region $-\frac{1}{2} < \operatorname{Re} s < \frac{1}{2}$ given by

$$\begin{aligned}
&\frac{(4\pi)^k}{4\pi i}\int_{(0)} G(s,w,z)\mathcal{Z}(s,w,z)\langle \mathcal{V}_{f,\overline{g}}, E(\cdot,\tfrac{1}{2}-\overline{z})\rangle dz \\
&\quad + (4\pi)^k \operatorname*{Res}_{z=\frac{3}{2}-s-w} G(s,w,z)\mathcal{Z}(s,w,z)\langle \mathcal{V}_{f,\overline{g}}, E(\cdot,\tfrac{1}{2}-\overline{z})\rangle \qquad (4.10)\\
&\quad + (4\pi)^k \operatorname*{Res}_{z=\frac{1}{2}-s} G(s,w,z)\mathcal{Z}(s,w,z)\langle \mathcal{V}_{f,\overline{g}}, E(\cdot,\tfrac{1}{2}-\overline{z})\rangle. \qquad (4.11)
\end{aligned}$$

We iterate this argument, as in Section 4 of [HHR13, p. 481-483]. Somewhat more specifically, when $\operatorname{Re}(s)$ approaches a negative half-integer, $\frac{1}{2}-n$, we can shift the line of integration for z right past the pole due to $G(s,z)$ at $z = s-\frac{1}{2}+n$, move s left past the line $\operatorname{Re}(s) = \frac{1}{2}-n$ and then shift the line of integration for z left, back to zero and over the pole at $z = \frac{1}{2}-s-n$. This gives meromorphic continuation of (4.8) arbitrarily far to the left, accumulating an additional pair of residual terms each time $\operatorname{Re} s$ passes a half-integer, and of a similar form to the first residual term coming from $G(s,z)$, appearing in (4.11).

We now specialize to $w = 0$. It is advantageous to codify some terminology for these additional terms appearing in the meromorphic continuation of the integral, i.e. terms like (4.10) and (4.11). We call these terms *residual* terms, as they come from residues in the z variable; these are distinct from residues in s, as these residual terms are functions in s. We also introduce a notation for these residual terms. Substituting $w = 0$ into (4.10), we get the residual term

$$\rho_{\frac{3}{2}}(s) = \frac{(4\pi)^k \zeta(2s-2)\Gamma(2s-2)}{\Gamma(s)\Gamma(s+k-1)\zeta^*(2s-2)}\langle \mathcal{V}_{f,\overline{g}}, \overline{E(\cdot,2-s)}\rangle. \qquad (4.12)$$

This residual term is distinguished as the only residual term appearing as a residue of the zeta functions.

The remaining residual terms all come from residues of Gamma functions, and all have a similar form. For each $m \geq 1$, there is a residual term $\rho_{\frac{3}{2}-m}(s)$ appearing for $\operatorname{Re} s < \frac{3}{2} - m$, appearing from an apparent pole of Gamma functions in $G(s,z)$ at $z = \frac{3}{2} - m - s$, given by

$$\begin{aligned}\rho_{\frac{3}{2}-m}(s) =& \frac{(-1)^{m-1}(4\pi)^k \zeta(1-m)\zeta(2s+m-2)\Gamma(2s+m-2)}{\Gamma(m)\Gamma(s)\Gamma(s+k-1)\zeta^*(4-2s-2m)} \times \\ & \langle \mathcal{V}_{f,\overline{g}}, \overline{E(\cdot, s+m-1)} \rangle.\end{aligned} \tag{4.13}$$

We summarize these computations with the following proposition.

Proposition 3.4.6. *The continuous spectrum component of $Z(s,0,f\times\overline{g})$, as given by (4.8) in Proposition 3.4.2, has meromorphic continuation to the complex plane. Further, the meromorphic continuation can be written explicitly as*

$$\frac{(4\pi)^k}{4\pi i}\int_{(0)} G(s,z)\mathcal{Z}(s,0,z)\langle \mathcal{V}_{f,\overline{g}}, \overline{E(\cdot,\tfrac{1}{2}-z)}\rangle dz + \sum_{0\leq m<\frac{3}{2}-\operatorname{Re} s} \rho_{\frac{3}{2}-m}(s),$$

where each residual term $\rho_{\frac{3}{2}-m}(s)$ is given by (4.12) (in the case that $m=0$) or (4.13) (when $m \geq 1$), and $\rho_{\frac{3}{2}-m}$ appears only when $\operatorname{Re} s < \frac{3}{2} - m$.

3.4.1 Polar Analysis of $Z(s,0,f\times\overline{g})$

Comparing the meromorphic continuations of the discrete spectrum component and continuous spectrum component of $Z(s,0,f\times\overline{g})$ reveals that the rightmost pole of $Z(s,0,f\times\overline{g})$ occurs in the first residual term, $\rho_{\frac{3}{2}}(s)$, appearing in (4.12). The pole occurs at $s=1$, caused by the Eisenstein series. By comparison, the discrete spectrum component is analytic for $\operatorname{Re} s > -\frac{1}{2} \pm it_j$, and the rest of the continuous spectrum is analytic for $\operatorname{Re} s > \frac{1}{2}$.

The residue at this pole is given by

$$\operatorname*{Res}_{s=1} \rho_{\frac{3}{2}}(s) = \operatorname*{Res}_{s=1} \frac{(4\pi)^k \zeta(2s-2)\Gamma(2s-2)}{\zeta^*(2s-2)\Gamma(s)\Gamma(s+k-1)} \langle \mathcal{V}_{f,\overline{g}}, E(\cdot, 2-\overline{s})\rangle.$$

Expand $\zeta^*(2s-2) = \zeta(2s-2)\Gamma(s-1)\pi^{1-s}$ in the denominator, cancel the two zeta functions, and use the Gamma duplication identity

$$\frac{\Gamma(2z)}{\Gamma(z)} = \frac{\Gamma(z+\frac{1}{2})2^{2z-1}}{\sqrt{\pi}}$$

with $z = s - 1$ to rewrite the residue as

$$\begin{aligned}\operatorname*{Res}_{s=1} \rho_{\frac{3}{2}}(s) &= \operatorname*{Res}_{s=1} \frac{\Gamma(s-\frac{1}{2})2^{2s-3}}{\sqrt{\pi}} \frac{\pi^{s-1}}{\Gamma(s)} \frac{(4\pi)^k}{\Gamma(s+k-1)} \langle \mathcal{V}_{f,\overline{g}}, E(\cdot, 2-\overline{s})\rangle \\ &= -\frac{(4\pi)^k}{\Gamma(k)} \operatorname*{Res}_{s=1} \langle f\overline{g}\, \mathrm{Im}(\cdot)^k, E(\cdot, \overline{s})\rangle.\end{aligned}$$

Through the relationship between the Eisenstein series and the Rankin–Selberg L-function (cf. §3.2.1), we can rewrite this as

$$\operatorname*{Res}_{s=1} \rho_{\frac{3}{2}}(s) = -\operatorname*{Res}_{s=1} \frac{L(s, f\times\overline{g})}{\zeta(2)}. \tag{4.14}$$

The next pole of $Z(s, 0, f \times \overline{g})$ also comes from the first residual term $\rho_{\frac{3}{2}}(s)$, occurring at $s = \frac{1}{2}$ from the Gamma function in the numerator. Similar to the computation of the residue at $s = 1$, we expand ζ^*, cancel the two zeta functions, and apply the Gamma duplication identity to recognize the residue as

$$\begin{aligned}\operatorname*{Res}_{s=\frac{1}{2}} \rho_{\frac{3}{2}}(s) &= \operatorname*{Res}_{s=\frac{1}{2}} \frac{(4\pi)^{s+k-1}}{2\sqrt{\pi}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)\Gamma(s+k-1)} \langle \mathcal{V}_{f,\overline{g}}, E(\cdot, 2-\overline{s})\rangle \\ &= \frac{1}{2\pi} \frac{(4\pi)^{k-\frac{1}{2}}}{\Gamma(k-\frac{1}{2})} \langle \mathcal{V}_{f,\overline{g}}, E(\cdot, \tfrac{3}{2})\rangle.\end{aligned}$$

We rewrite this as a special value of the Rankin–Selberg L-function,

$$\operatorname*{Res}_{s=\frac{1}{2}} \rho_{\frac{3}{2}}(s) = \frac{1}{2\pi} \frac{(k-\frac{1}{2})}{4\pi} \frac{(4\pi)^{k+\frac{1}{2}}}{\Gamma(k+\frac{1}{2})} \langle \mathcal{V}_{f,\overline{g}}, E(\cdot, \tfrac{3}{2})\rangle = \frac{(k-\frac{1}{2})}{4\pi^2} \frac{L(\frac{3}{2}, f\times\overline{g})}{\zeta(3)}. \tag{4.15}$$

Returning to the rest of the meromorphic continuation of the continuous spectrum, let us examine the second residual term $\rho_{\frac{3}{2}-1}(s) = \rho_{\frac{1}{2}}(s)$, which appears as part of the meromorphic continuation only for $\operatorname{Re} s < \frac{1}{2}$. We simplify the expression in (4.13), with $m = 1$,

$$\rho_{\frac{1}{2}}(s) = \frac{(4\pi)^k \zeta(0)}{\Gamma(s+k-1)} \frac{\zeta(2s-1)}{\zeta^*(2-2s)} \frac{\Gamma(2s-1)}{\Gamma(s)} \langle \mathcal{V}_{f,\overline{g}}, E(\cdot, \overline{s})\rangle.$$

By using the functional equation for $\zeta^*(2-2s)$, the Gamma duplication formula, and recognizing the Eisenstein series inner product as a sum of two Rankin–Selberg L-functions, this simplifies to

$$\rho_{\frac{1}{2}}(s) = -\frac{L(s, f\times\overline{g})}{\zeta(2s)}. \tag{4.16}$$

We now recognize that $\rho_{\frac{1}{2}}(s)$ has poles at zeroes of $\zeta(2s)$.

Remark 3.4.7. At first glance, it would appear that $\rho_{\frac{1}{2}}(s)$ also has a pole at $s = 1$, coming from the pole of the Rankin–Selberg convolution, and that this term therefore contributes a pole to $Z(s, 0, f \times \overline{g})$ at $s = 1$. However, the term $\rho_{\frac{1}{2}}(s)$ does not appear as part of the meromorphic continuation of $Z(s, 0, f \times \overline{g})$ except when $\operatorname{Re} s < \frac{1}{2}$, so this term does not contribute a pole at $s = 1$.

More generally, for each $m \geq 1$, the residual term $\rho_{\frac{3}{2}-m}(s)$, which appears for $\operatorname{Re} s < \frac{3}{2} - m$, also contributes poles. As in (4.16), the Eisenstein series in $\rho_{\frac{3}{2}-m}(s)$ introduces poles at $s = \frac{\gamma}{2} - m + 1$ for each nontrivial zero γ of $\zeta(s)$, in addition to potential poles at negative integers and half-integers from the Gamma function in the numerator.

Poles appearing in the discrete spectrum component do not exhibit the same properties of cancellation, aside from those noted in Lemmas 3.4.4 and 3.4.5. Analysis of the Gamma functions in the discrete component, in (4.7), shows that there are potential simple poles at $s = \frac{1}{2} \pm it_j - n$ for $n \in \mathbb{Z}_{\geq 0}$. The Lemmas 3.4.4 and 3.4.5 show that those poles occurring at $s = \frac{1}{2} \pm it_j - n$ with n even are cancelled by trivial zeroes. Together, these indicate that there are potential poles at $s = \frac{1}{2} \pm it_j - n$ for each odd, positive integer n.

3.4.2 Analytic Behavior of $W(s; f, \overline{g})$

Recall that

$$W(s; f, \overline{g}) = \frac{L(s, f \times \overline{g})}{\zeta(2s)} + Z(s, 0, f \times \overline{g}).$$

As noted in (4.14), the leading pole of $L(s, f \times \overline{g})\zeta(2s)^{-1}$ perfectly cancels with the leading pole of $Z(s, 0, f \times \overline{g})$. Therefore $W(s, f, \overline{g})$ is analytic for $\operatorname{Re} s > \frac{1}{2}$ and has a pole at $s = \frac{1}{2}$, identified in (4.15).

Further, the second residual term, $\rho_{\frac{1}{2}}(s)$, was shown to be exactly $-L(s, f \times \overline{g})\zeta(2s)^{-1}$ in (4.16), and appears for $\operatorname{Re} s < \frac{1}{2}$. Therefore the Rankin–Selberg L-function $L(s, f \times \overline{g})\zeta(2s)^{-1}$ perfectly cancels with $\rho_{\frac{1}{2}}(s)$ for $\operatorname{Re} s < \frac{1}{2}$. For $\operatorname{Re} s < \frac{1}{2}$, the analytic behavior of $W(s; f, \overline{g})$ is determined entirely by the analytic behavior of $Z(s, 0, f \times \overline{g})$ (and omitting $\rho_{\frac{1}{2}}(s)$).

Therefore $W(s; f, \overline{g})$ has meromorphic continuation to $\mathbb{C}$. For $\operatorname{Re} s > -\frac{1}{2}$, the only possible poles of $W(s; f, \overline{g})$ are at $s = \frac{1}{2}$, coming from (4.15), and those at $s = -\frac{1}{2} \pm it_j$ coming from exceptional eigenvalues of the discrete spectrum. (There are no exceptional eigenvalues on $\mathrm{SL}(2, \mathbb{Z})$). Collecting the analytic data, we have proved the following.

Theorem 3.4.8. *Let f, g be two holomorphic cusp forms on $\mathrm{SL}(2,\mathbb{Z})$. Maintaining the same notation as above, the function $W(s; f, \overline{g})$ has a meromorphic continuation to $\mathbb{C}$ given by the Rankin–Selberg L-function (2.1) and spectral decomposition in Proposition 3.4.2, with potential poles at s with $\operatorname{Re} s \leq \frac{1}{2}$ and $s \in \mathbb{Z} \cup (\mathbb{Z} + \frac{1}{2}) \cup \mathfrak{S} \cup \mathfrak{Z}$, where $\mathfrak{Z}$ denotes the set of shifted zeta-zeroes $\{-1 + \frac{\gamma}{2} - n : n \in \mathbb{Z}_{\geq 0}\}$, and $\mathfrak{S}$ denotes the set of shifted discrete types $\{-\frac{1}{2} \pm it_j - n : n \in \mathbb{Z}_{\geq 0}, n \text{ odd}\}$.*

The leading pole is at $s = \frac{1}{2}$ and

$$\operatorname*{Res}_{s=\frac{1}{2}} W(s; f, \overline{g}) = \frac{(k-\frac{1}{2})}{4\pi^2} \frac{L(\frac{3}{2}, f \times \overline{g})}{\zeta(3)}.$$

3.4.3 Complete Meromorphic Continuation of $D(s, S_f \times \overline{S_g})$

With Theorem 3.4.8 and the decomposition from Proposition 3.3.4, we can quickly give the meromorphic continuation of the Dirichlet series $D(s, S_f \times \overline{S_g})$. In particular, by Proposition 3.3.4, we know that

$$D(s, S_f \times \overline{S_g}) = W(s; f, \overline{g}) + \frac{1}{2\pi i} \int_{(\gamma)} W(s-z; f, \overline{g}) \zeta(z) \frac{\Gamma(z)\Gamma(s-z+k-1)}{\Gamma(s+k-1)}\, dz,$$

where initially $\operatorname{Re} s$ is large and $\gamma \in (1, \operatorname{Re}(s) - 1)$. Notice that $W(s; f, \overline{g})$ can also be written as a single Dirichlet series as

$$\begin{aligned} W(s; f, \overline{g}) &= \frac{L(s, f \times \overline{g})}{\zeta(2s)} + Z(s, 0, f \times \overline{g}) \\ &= \sum_{n,h \geq 1} \frac{a(n)\overline{b(n)} + a(n)\overline{b(n-h)} + a(n-h)\overline{b(n)}}{n^{s+k-1}} \\ &= \sum_{\substack{n \geq 1 \\ h \geq 0}} \frac{a(n)\overline{b(n-h)} + a(n-h)\overline{b(n)} - a(n)\overline{b(n)}}{n^{s+k-1}} \\ &= \sum_{n \geq 1} \frac{a(n)\overline{S_g(n)} + S_f(n)\overline{b(n)}}{n^{s+k-1}} =: \sum_{n \geq 1} \frac{w(n)}{n^{s+k-1}}. \end{aligned}$$

We denote the nth coefficient of this Dirichlet series of $w(n)$. As $a(n) \ll n^{\frac{k-1}{2}+\epsilon}$ and $S_f(n) \ll n^{\frac{k-1}{2}+\frac{1}{3}}$, we know that $w(n) \ll n^{k-1+\frac{1}{3}+\epsilon}$. Thus $W(s; f, \overline{g})$ converges absolutely for $\operatorname{Re} s > \frac{4}{3}$.

Consider $D(s, S_f \times \overline{S_g})$ for $\operatorname{Re} s > 4$ and $\gamma = 2$ initially, so that both $W(s; f, \overline{g})$ and $W(s-z; f, \overline{g})$ are absolutely convergent. Shifting the line of z integration to $-M - \frac{1}{2}$ for some positive integer M passes several poles occurring when $z = 1$ (from

$\zeta(z))$ or $z = -j$ with $j \in \mathbb{Z}_{\geq 0}$ (from $\Gamma(z)$). Notice that in this region, $W(s-z; f, \overline{g})$ converges absolutely, $\zeta(z)$ has at most polynomial growth in vertical strips, and the Gamma functions have exponential decay for any fixed s. By Cauchy's Theorem, we have

$$\begin{aligned}
&D(s, S_f \times \overline{S_g}) = \\
&= W(s; f, \overline{g}) + \sum_{-M \leq j \leq 1} \operatorname*{Res}_{z=j} W(s-z; f, \overline{g}) \zeta(z) \frac{\Gamma(z)\Gamma(s-z+k-1)}{\Gamma(s+k-1)} \\
&\quad + \frac{1}{2\pi i} \int_{(-M-\frac{1}{2})} W(s-z; f, \overline{g}) \zeta(z) \frac{\Gamma(z)\Gamma(s-z+k-1)}{\Gamma(s+k-1)} \, dz \\
&= W(s; f, \overline{g}) + \frac{W(s-1; f, \overline{g})}{s+k-2} + \sum_{j=0}^{M} \frac{(-1)^j}{j!} W(s+j; f, \overline{g}) \zeta(-j) \frac{\Gamma(s+j+k-1)}{\Gamma(s+k-1)} \\
&\quad + \frac{1}{2\pi i} \int_{(-M-\frac{1}{2})} W(s-z; f, \overline{g}) \zeta(z) \frac{\Gamma(z)\Gamma(s-z+k-1)}{\Gamma(s+k-1)} \, dz.
\end{aligned}$$

Each of the residues gives an expression containing $W(s; f, \overline{g})$ with clear meromorphic continuation to the plane. The remaining shifted integral contains $W(s-z; f, \overline{g})$ in its integrand, with $\operatorname{Re} z = -M - \frac{1}{2}$. Therefore $\operatorname{Re} s - z = \operatorname{Re} s + M + \frac{1}{2}$, and so $W(s-z; f, \overline{g})$ is absolutely convergent for $\operatorname{Re} s > -M + \frac{5}{2}$. As $\zeta(z)$ has only polynomial growth and $\Gamma(z)\Gamma(s-z+k-1)$ has exponential decay in vertical strips, we see that the integral represents an analytic function of s for $\operatorname{Re} s > -M + \frac{5}{2}$. Therefore the entire right hand side has meromorphic continuation to the region $\operatorname{Re} s > -M + \frac{5}{2}$. As M is arbitrary, we have proved the following, which we record as a corollary to Theorem 3.4.8.

Corollary 3.4.9. *The Dirichlet series $D(s, S_f \times \overline{S_g})$ has meromorphic continuation to the entire complex plane.*

Remark 3.4.10. Very similar work gives the meromorphic continuation for $D(s, S_f \times \overline{S_g})$, mainly replacing $\overline{g}$ with $T_{-1}g$ in the above formulation. This distinction only matters at higher levels when f and g have nontrivial nebentypus, and the spectral expansion is modified accordingly.

Analysis of the exact nature of the poles of $D(s, S_f \times \overline{S_g})$ can be performed directly on this presentation of the meromorphic continuation. In many cases, the leading behavior of integral transforms on $D(s, S_f \times \overline{S_g})$ will come from $W(s-1; f, \overline{g})/(s+k-2)$, as this term contains the largest negative shift in s. For later reference, it will be useful to view $D(s, S_f \times \overline{S_g})$ in this form for clear arithmetic application. We codify this in the following lemma.

Lemma 3.4.11.

$$D(s, S_f \times \overline{S_g}) = W(s; f, \overline{g}) + \frac{W(s-1; f, \overline{g})}{s+k-2} + \sum_{j=0}^{M} \frac{(-1)^j}{j!} W(s+j; f, \overline{g}) \zeta(-j) \frac{\Gamma(s+j+k-1)}{\Gamma(s+k-1)} + \frac{1}{2\pi i} \int_{(-M-\frac{1}{2})} W(s-z; f, \overline{g}) \zeta(z) \frac{\Gamma(z)\Gamma(s-z+k-1)}{\Gamma(s+k-1)}\, dz.$$

3.5 Second-Moment Analysis

It is now necessary to estimate the growth of $D(s, S_f \times \overline{S_g})$ and to use the analytic properties described above to study the sizes of sums of coefficients of cusp forms. It will be necessary to understand the size of growth of $D(s, S_f \times \overline{S_g})$, but it is relatively straightforward to see that $D(s, S_f \times \overline{S_g})$ has polynomial growth in vertical strips.

Lemma 3.5.1. *For $\sigma < \operatorname{Re} s < \sigma'$ and s uniformly away from poles, there exists some A such that*

$$D(s, S_f \times \overline{S_g}) \ll |\operatorname{Im} s|^A.$$

Therefore $D(s, S_f \times \overline{S_g})$ is of polynomial growth in vertical strips.

Proof. From Lemma 3.4.11 it is only necessary to study the growth properties of $W(s; f, \overline{g})$ and the Mellin-Barnes integral transform of $W(s; f, \overline{g})$.

We first handle $W(s; f, \overline{g})$. The diagonal component of $W(s; f, \overline{g})$ is just the Rankin–Selberg L-function $L(s, f \times \overline{g})\zeta(2s)^{-1}$, which has polynomial growth in vertical strips as a consequence of the Phragmén-Lindelöf convexity principle and the functional equation.

As noted in Remark 3.4.3, the discrete spectrum and integral term in the continuous spectrum each have polynomial growth in vertical strips. It remains to consider the possible contribution from the residual terms $\rho_{\frac{3}{2}}(s)$ and $\rho_{\frac{3}{2}-m}(s)$. These each consist of a product of zeta functions, Gamma functions, and Rankin–Selberg L-functions, and a quick analysis through Stirling's approximation shows that the exponential contributions from the Gamma functions all perfectly cancel. Therefore these are also of polynomial growth.

We now handle the Mellin-Barnes transform of $W(s; f, \overline{g})$. We actually prove a slightly more general result.

Let $F(s)$ be a function of polynomial growth in $|\mathrm{Im}\, s|$ in vertical strips containing σ. Then the function

$$\frac{1}{2\pi i}\int_{(\sigma)} F(s-z)\zeta(z)\frac{\Gamma(z)\Gamma(s-z)}{\Gamma(s)}dz$$

has at most polynomial growth in $|\mathrm{Im}\, s|$. Indeed, through Stirling's Approximation, the integrand is bounded by

$$|\mathrm{Im}\, s|^A|\mathrm{Im}(s-z)|^B|\mathrm{Im}\, z|^C \exp\bigg(-\frac{\pi}{2}\Big(|\mathrm{Im}\, z|+|\mathrm{Im}(s-z)|-|\mathrm{Im}\, s|\Big)\bigg).$$

Therefore, for $|\mathrm{Im}\, z| > |\mathrm{Im}\, s|$, the integrand has exponential decay and converges rapidly. Thus the integral is essentially of an integrand of polynomial growth along an interval of length $2|\mathrm{Im}\, s|$, leading to an overall polynomial bound in $|\mathrm{Im}\, s|$. □

This is already sufficient for many applications. Consider the integral transform

$$\frac{1}{2\pi i}\int_{(\sigma)} D(s, S_f\times\overline{S_g})X^s\Gamma(s)ds = \sum_{n\geq 1}\frac{S_f(n)\overline{S_g(n)}}{n^{k-1}}e^{-n/X}, \tag{5.1}$$

as described in §2.3. Initially take σ large enough to be in the domain of absolute convergence of $D(s, S_f\times\overline{S_g})$, say $\sigma\geq 4$.

Through Lemma 3.4.11, we rewrite (5.1) as

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(4)}\left(\frac{1}{2}W(s;f,\overline{g})+\frac{W(s-1;f,\overline{g})}{s+k-2}\right)X^s\Gamma(s)ds\\ &\quad+\frac{1}{(2\pi i)^2}\int_{(4)}\int_{(-1+\epsilon)} W(s-z;f,\overline{g})\zeta(z)\frac{\Gamma(z)\Gamma(s-z+k-1)}{\Gamma(s+k-1)}dzX^s\Gamma(s)ds.\end{aligned} \tag{5.2}$$

From the proof and statement of Theorem 3.4.8, we see that $W(s;f,\overline{g})$ is analytic in $\mathrm{Re}\, s > -\frac{1}{2}$ except for poles at $s=\frac{1}{2}$ and at $s=-\frac{1}{2}\pm it_j$. Therefore when we shift lines of s-integration in (5.2) to $\frac{1}{2}+2\epsilon$ passes a pole at $s=\frac{3}{2}$ from $W(s-1;f,\overline{g})$, and otherwise no poles.

Remark 3.5.2. For general level, we shift lines of s-integration to $\frac{1}{2}+\theta+2\epsilon$, where $\theta<\frac{7}{64}$ is the best-known progress towards Selberg's Eigenvalue Conjecture, as noted above.

By Lemma 3.5.1, this shift is justified and the resulting integral converges absolutely. Therefore

$$\sum_{n\geq 1}\frac{S_f(n)\overline{S_g(n)}}{n^{k-1}}e^{-n/X} = CX^{\frac{3}{2}}+O_{f,g,\epsilon}(X^{\frac{-1}{2}+\theta+\epsilon})$$

We can evaluate the residue C using Theorem 3.4.8. Note that the same analysis holds on $D(s, S_f\times S_g)$ as well. In total, we have proved the following theorem.

Theorem 3.5.3. *Suppose f and g are weight k holomorphic cusp forms on $\mathrm{SL}(2,\mathbb{Z})$. For any $\epsilon > 0$,*

$$\sum_{n\geq 1} \frac{S_f(n)\overline{S_g(n)}}{n^{k-1}} e^{-n/X} = CX^{\frac{3}{2}} + O_{f,g,\epsilon}(X^{\frac{1}{2}+\epsilon})$$
$$\sum_{n\geq 1} \frac{S_f(n)S_g(n)}{n^{k-1}} e^{-n/X} = C'X^{\frac{3}{2}} + O_{f,g,\epsilon}(X^{\frac{1}{2}+\epsilon})$$

where

$$C = \frac{\Gamma(\frac{3}{2})}{4\pi^2}\frac{L(\frac{3}{2}, f\times\overline{g})}{\zeta(3)}, \qquad C' = \frac{\Gamma(\frac{3}{2})}{4\pi^2}\frac{L(\frac{3}{2}, f\times g)}{\zeta(3)}.$$

As an immediate corollary, we have the following smoothed analogue of the Classical Conjecture.

Corollary 3.5.4.

$$\sum_{n\geq 1} \frac{|S_f(n)|^2}{n^{k-1}} e^{-n/X} = CX^{\frac{3}{2}} + O_{f,\epsilon}(X^{\frac{1}{2}+\epsilon}),$$

where C is the special value of $L(\frac{3}{2}, f\times\overline{f})\Gamma(\frac{3}{2})(\zeta(3)4\pi^2)^{-1}$ as above.

3.6 A General Cancellation Principle

While the techniques and methodology employed so far should work for general weight and level, it is not immediately obvious that that the miraculous cancellation that occurs in the level 1 case should always occur. In particular, it is not clear that the continuous spectrum of $Z(s,0,f\times\overline{g})$ will always perfectly cancel both the leading pole and potentially infinitely many poles from the zeta zeroes of $L(s,f\times g)\zeta(2s)^{-1}$.

In the case when $f = g$ are the same cusp form, we can compare our methodology with the results of Chandrasekharan and Narasimhan to show that the leading polar cancellation does always occur. A more detailed analysis using the same methodology as the rest of this chapter would likely be able to show general cancellation.

Remark 3.6.1. In Section 6 of the soon-to-be-published paper [HKLDW17a], my collaborators and I explicitly show that this cancellation continues to hold for cusp forms f and g on $\Gamma_0(N)$ when N is square-free. This is much stronger than what is showed in the rest of this section concerning general cancellation between the diagonal and off-diagonal sums corresponding to $f\times\overline{f}$.

Suppose $f(z) = \sum a(n)e(nz)$ is a cusp form on $\Gamma_0(N)$ and of weight $k \in \mathbb{Z} \cup (\mathbb{Z}+\frac{1}{2})$ with $k > 2$. Theorem 1 of [CN64] gives that

$$\frac{1}{X}\sum_{n\leq X}\frac{|S_f(n)|^2}{n^{k-1}} = CX^{\frac{1}{2}} + O(\log^2 X). \tag{6.1}$$

Performing the decomposition from Proposition 3.3.4 leads us to again study $Z(s, 0, f \times \overline{f})$ and $W(s; f, \overline{f})$. The Rankin–Selberg convolution $L(s, f \times f)/\zeta(2s)$ has a pole at $s = 1$. This pole must cancel with poles from $Z(s, 0, f \times \overline{f})$, as otherwise the methodology of this chapter contradicts (6.1). Stated differently, we must have that the leading contribution of the diagonal term cancels perfectly with a leading contribution from the off-diagonal,

$$\operatorname*{Res}_{s=1}\sum_{n\geq 1}\frac{|a(n)|^2}{n^{s+k-1}} = -\operatorname*{Res}_{s=1}\sum_{n,h\geq 1}\frac{a(n)\overline{a(n-h)} + \overline{a(n)}a(n-h)}{n^{s+k-1}}.$$

We investigate this cancellation further by sketching the arguments of §3.4 and §3.5 in greater generality.

The spectral decomposition corresponding to Proposition 3.4.2 is more complicated since we must now use the Selberg Poincaré series on $\Gamma_0(N)$

$$P_h(z, s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \operatorname{Im}(\gamma z)^s e(h\gamma \cdot z).$$

The spectral decomposition of P_h will involve Eisenstein series associated to each cusp $\mathfrak{a}$ of $\Gamma_0(N)$. These Eisenstein series have expansions

$$E_{\mathfrak{a}}(z, w) = \delta_{\mathfrak{a}} y^w + \varphi_{\mathfrak{a}}(0, w)y^{1-w} + \sum_{m\neq 0}\varphi_{\mathfrak{a}}(m, w)W_w(|m|z),$$

where $\delta_{\mathfrak{a}} = 1$ if $\mathfrak{a} = \infty$ and is 0 otherwise,

$$\varphi(0, w) = \sqrt{\pi}\frac{\Gamma(w-\frac{1}{2})}{\Gamma(w)}\sum_c c^{-2w}S_{\mathfrak{a}}(0, 0; c)$$
$$\varphi(m, w) = \frac{\pi^w}{\Gamma(w)}|m|^{w-1}\sum_c c^{-2w}S_{\mathfrak{a}}(0, m; c)$$

are generalized Whittaker-Fourier coefficients,

$$W_w(z) = 2\sqrt{y}K_{w-\frac{1}{2}}(2\pi y)e(x)$$

is a Whittaker function, $K_\nu(z)$ is a K-Bessel function, and

$$S_{\mathfrak{a}}(m, n; c) = \sum_{\left(\begin{smallmatrix} a & \cdot \\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\sigma_\alpha^{-1}\Gamma_0(N)/\Gamma_\infty} e\left(m\frac{d}{c} + n\frac{a}{c}\right)$$

is a Kloosterman sum associated to double cosets of $\Gamma_0(N)$ with

$$\Gamma_\infty = \left\langle \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} : n \in \mathbb{Z} \right\rangle \subset \mathrm{SL}_2(\mathbb{Z}).$$

This expansion is given in Theorem 3.4 of [Iwa02].

Letting μ_j be an orthonormal basis of the residual and cuspidal spaces, we may expand $P_h(z,s)$ by the Spectral Theorem (as presented in Theorem 15.5 of [IK04]) to get

$$\begin{aligned} P_h(z,s) = &\sum_j \langle P_h(\cdot,s), \mu_j\rangle \mu_j(z) \\ &+ \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{\mathbb{R}} \langle P_h(\cdot,s), E_{\mathfrak{a}}(\cdot, \tfrac{1}{2}+it)\rangle E_{\mathfrak{a}}(z, \tfrac{1}{2}+it)\, dt. \end{aligned} \tag{6.2}$$

This is more complicated than the $\mathrm{SL}_2(\mathbb{Z})$ spectral expansion in (4.2) for two major reasons: we are summing over cusps and the Kloosterman sums within the Eisenstein series are trickier to handle. Continuing as before, we try to understand the shifted convolution sum

$$Z(s,w,f\times f) = \frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)} \sum_{h\geq 1} \frac{\langle |f|^2 \operatorname{Im}(\cdot)^k, P_h\rangle}{h^w}$$

by substituting the spectral expansion for $P_h(z,s)$ and producing a meromorphic continuation.

The analysis of the discrete spectrum is almost exactly the same: it is analytic for $\operatorname{Re} s > -\frac{1}{2}+\theta$. The only new facet is understanding the continuous spectrum component corresponding to (6.2). We expect that the continuous spectrum of $Z(s,0,f\times\overline{f})$ has leading poles that perfectly cancel the leading pole of $L(s,f\times\overline{f})\zeta(2s)^{-1}$.

Using analogous methods to those in Section 3.4, we compute the continuous spectrum of $Z(s,0,f\times\overline{f})$ to get

$$\begin{aligned} &\sum_{h\geq 1} \frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)} \sum_{\mathfrak{a}} \frac{1}{4\pi i} \int_{(\frac{1}{2})} \langle P_h(\cdot,s), E_{\mathfrak{a}}(\cdot,t)\rangle \langle |f|^2 \operatorname{Im}(\cdot)^k, \overline{E_{\mathfrak{a}}(\cdot,t)}\rangle\, dt \\ &= \frac{(4\pi)^k}{\Gamma(s+k-1)\Gamma(s)} \sum_{\mathfrak{a}} \int_{-\infty}^{\infty} \left(\sum_{h,c\geq 1} \frac{S_{\mathfrak{a}}(0,h;c)}{h^{s+it}c^{1-2it}} \frac{\pi^{\frac{1}{2}-it}}{\Gamma(\frac{1}{2}-it)} \right) \times \\ &\quad \times \Gamma(s-\tfrac{1}{2}+it)\Gamma(s-\tfrac{1}{2}-it) \langle |f|^2 \operatorname{Im}(\cdot)^k, \overline{E_{\mathfrak{a}}(\cdot,\tfrac{1}{2}+it)}\rangle\, dt. \end{aligned}$$

We've placed parentheses around the arithmetic part, including the Kloosterman sums and factors for completing a zeta function that appears within the Kloosterman sums.

The arithmetic part of the Eisenstein series are classically-studied L-functions, and each satisfies analogous analytic properties to the function denoted $\mathcal{Z}(s,0,z)$ in §3.4. We summarize the results of this section with the following theorem.

Theorem 3.6.2. *Let f be a weight $k>2$ cusp form on $\Gamma_0(N)$. Then*

$$\operatorname*{Res}_{s=1}\sum_{n\geq 1}\frac{|a(n)|^2}{n^{s+k-1}} = -\operatorname*{Res}_{s=1}\sum_{n,h\geq 1}\frac{a(n)\overline{a(n-h)}}{n^{s+k-1}} - \operatorname*{Res}_{s=1}\sum_{n,h\geq 1}\frac{\overline{a(n)}a(n-h)}{n^{s+k-1}},$$

or equivalently

$$\begin{aligned}
-\frac{1}{2}\operatorname*{Res}_{s=1}\frac{L(s,f\times f)}{\zeta(2s)} &= \\
&= \operatorname*{Res}_{s=1}\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{\mathbb{R}}\langle P_h(\cdot,s),E_{\mathfrak{a}}(\cdot,\tfrac{1}{2}+it)\rangle\langle |f|^2\operatorname{Im}(\cdot)^k,\overline{E_{\mathfrak{a}}(\cdot,\tfrac{1}{2}+it)}\rangle\,dt \\
&= \operatorname*{Res}_{s=1}\frac{(4\pi)^k}{\Gamma(s+k-1)\Gamma(s)}\sum_{\mathfrak{a}}\int_{-\infty}^{\infty}\left(\sum_{h,c\geq 1}\frac{S_{\mathfrak{a}}(0,h;c)}{h^{s+it}c^{1-2it}}\frac{\pi^{\frac{1}{2}-it}}{\Gamma(\frac{1}{2}-it)}\right) \\
&\quad\times\Gamma(s-\tfrac{1}{2}+it)\Gamma(s-\tfrac{1}{2}-it)\langle |f|^2\operatorname{Im}(\cdot)^k,\overline{E_{\mathfrak{a}}(\cdot,\tfrac{1}{2}+it)}\rangle\,dt.
\end{aligned}$$

Chapter Four

Applications of Dirichlet Series of Sums of Coefficients

In this chapter, we highlight some applications of the Dirichlet series $D(s, S_f \times \overline{S_g})$ and $D(s, S_f \times S_g)$. We begin with completed applications, summarizing the results and methodology. Towards the end of the chapter, we highlight ongoing and future work.

Reacknowledgements to my Collaborators

Each of the applications in this chapter were carried out (or are being carried out) with Alex Walker, Chan Ieong Kuan, and Tom Hulse, each of whom are my academic brothers, collaborators, and friends.

4.1 Applications

The Dirichlet series $D(s, S_f \times \overline{S_g})$ and $D(s, S_f \times S_g)$ present a new avenue for investigating the behavior of S_f, $S_f(n)S_g(n)$, and related objects. As S_f is analogous to the error term in the Gauss Circle Problem (cf. Chapter 1), it is perhaps most natural to ask about the sizes of $S_f(n)$.

Long Sums

One result of this form was presented in §3, in Theorem 3.5.3 giving smoothed averages for $S_f(n)\overline{S_g(n)}$ and, if $f = g$, smoothed averages for $|S_f(n)|^2$. This theorem was proved completely for f and g on level 1.

In [HKLDW17a], my collaborators and I analyze further smoothed asymptotics for $S_f(n)\overline{S_g(n)}$ for forms f and g on $\Gamma_0(N)$ for N squarefree, proving analogous results to Theorem 3.5.3 for forms even of half-integral weight. When $f = g$, this is a generalized smoothed analogue to the result of Cramér [Cra22] giving that

$$\frac{1}{X}\int_0^X \Big|\sum_{n\leq t} r_2(n) - \pi t\Big|^2 dt = cX^{1/2} + O(X^{\frac{1}{4}+\epsilon}),$$

giving evidence towards a generalized Circle problem.

It is very interesting (and very new) that if $f \neq g$, then $S_f S_g$ still appears to satisfy a generalized Circle problem. Recalling Chandrasekharan and Narasimhan's result that $S_f(X) = \Omega_\pm(X^{\frac{k-1}{2}+\frac{1}{4}})$, it's apparent that both $S_f(n)$ and $S_g(n)$ oscillate in size between $\pm n^{\frac{k-1}{2}+\frac{1}{4}}$. From first principles alone, it seems possible that $S_f(n)$

might be small or negative with $S_g(n)$ is large and positive, leading to large cancellation in average sums $\sum S_f(n)S_g(n)$. But Theorem 3.5.3 indicates that there is not large cancellation of this sort. Indeed, Theorem 3.5.3 roughly indicates that the partial sums of Fourier coefficients of $f(z)$ correlate about as well with the partial sums of Fourier coefficients of $g(z)$ as with itself, up to the constant $L(\frac{3}{2}, f \times g)$ of proportionality.

By mimicking the techniques of Chapter 3, it is possible to apply different integral transforms to $D(s, S_f \times \overline{S_g})$ or $D(s, S_f \times S_g)$, either to get different long-average estimates, or estimates of a different variety.

Short-Interval Averages

Now restrict attention to f a full-integer weight cusp form on $\mathrm{SL}(2,\mathbb{Z})$, and suppose that $f = g$. In [HKLDW17c], my collaborators and I analyze short-interval estimates of the type

$$\frac{1}{X^{2/3}(\log X)^{1/6}} \sum_{|n-X|<X^{2/3}(\log X)^{1/6}} |S_f(n)|^2 \ll X^{k-1+\frac{1}{2}}, \tag{1.1}$$

which says essentially that the Classical Conjecture holds *on average* over short intervals of width $X^{\frac{2}{3}}(\log X)^{\frac{1}{6}}$ around X. This is qualitatively a much stronger result than the long-interval estimate, and is a vast improvement over the previous best result of this type due to Jarnik [Jut87],

$$\frac{1}{X^{\frac{3}{4}+\epsilon}} \sum_{|n-X|<X^{\frac{3}{4}+\epsilon}} |S_f(n)|^2 \ll X^{k-1+\frac{1}{2}}.$$

It is interesting to note that it is possible to get bounds for individual sums $S_f(n)$ from short-interval estimates.

Proposition 4.1.1. *Suppose that*

$$\frac{1}{X^w} \sum_{|n-X|<X^w} |S_f(n)|^2 \ll X^{k-1+\frac{1}{2}}$$

for $w \geq \frac{1}{4}$. *Then also*

$$S_f(X) \ll X^{\frac{k-1}{2}+\frac{1}{4}+(\frac{w}{3}-\frac{1}{12})}.$$

Proof. We only sketch the proof. For each individual Fourier coefficient $a(n)$, we have Deligne's bound $a(n) \ll n^{\frac{k-1}{2}}$. Suppose there is an X such that $S_f(X) \gg X^{\frac{k-1}{2}+\frac{1}{4}+\alpha}$.

Then $S_f(X+\ell) \gg X^{\frac{k-1}{2}+\frac{1}{4}+\alpha}$ for $\ell < X^{\frac{1}{4}+\alpha-\epsilon}$ for any $\epsilon > 0$, as it takes approximately $X^{\frac{1}{4}+\alpha}$ coefficients $a(n)$ to combine together to cancel $S_f(X)$. But then

$$\frac{1}{X^w} \sum_{|n-X|<X^w} |S_f(X)|^2 \gg \frac{1}{X^w} X^{k-1+\frac{1}{2}+2\alpha} X^{\min(w,\frac{1}{4}+\alpha)}.$$

The $X^{\min(w,\frac{1}{4}+\alpha)}$ term comes from the width of the interval where each $S_f(X+\ell)$ is approximately $X^{\frac{k-1}{2}+\frac{1}{4}+\alpha}$. Comparing exponents of X with

$$\frac{1}{X^w} \sum_{|n-X|<X^w} |S_f(n)|^2 \ll X^{k-1+\frac{1}{2}}$$

shows that

$$2\alpha + \min(w, \tfrac{1}{4}+\alpha) \leq w,$$

from which either $\alpha = 0$ or $\alpha < \frac{1}{3}(w - \frac{1}{4})$. □

Corollary 4.1.2. *A short-interval estimate of the type (1.1), with interval $|n-X| \leq X^{\frac{1}{4}+\epsilon}$, would prove the Classical Conjecture.*

The short-interval estimate (1.1) only produces the individual estimate $S_f(X) \ll X^{\frac{k-1}{2}+\frac{1}{4}+\frac{5}{36}}$, which is 1/18 worse than the current best-known individual bound $S_f(X) \ll X^{\frac{k-1}{2}+\frac{1}{3}}$. On the other hand, it is by far the strongest short-interval average estimate. (Note that the individual bound $S_f(X) \ll X^{\frac{k-1}{2}+\frac{1}{3}}$ doesn't give a Classical Conjecture on average type result for any interval length).

Remark 4.1.3. It is possible to use the short-interval estimate (1.1), along with an argument used in Chapter 5 which makes use of multiple integral transforms in combination, to recover the Hafner-Ivić tyle bound $S_f(n) \ll n^{\frac{k-1}{2}+\frac{1}{3}}$.

To prove the estimate (1.1), one builds upon the meromorphic information of $D(s, S_f \times \overline{S_f})$ and applies an integral transform

$$\frac{1}{2\pi i} \int_{(4)} D(s, S_f \times \overline{S_f}) \exp\left(\frac{\pi s^2}{y^2}\right) \frac{X^s}{y} ds$$

to understand (essentially) a sum over the interval $|n - X| < X/y$. For more details and analysis, refer to [HKLDW17c].

Sign-Changes of Sums of Coefficients of Cusp Forms

As noted in Chapter 8, the original guiding question that led to the investigation of $D(s, S_f \times \overline{S_g})$ was concerning the sign changes within the sequence $\{S_f(n)\}_{n\in\mathbb{N}}$.

In [HKLDW16], my collaborators and I succeeded in answering our original guiding question.[†]

In this paper, we proved a veritable cornucopia of results concerning the sign changes of coefficients and sums of coefficients of cusp forms on GL(2) and GL(3). Here, I emphasize what I focused on, and what follows most naturally from the considerations of Chapter 3.

Let f be a weight 0 Maass form or a holomorphic cusp form of full or half-integer weight k on a congruence subgroup $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$, possibly with nontrivial nebentypus. Write f as either

$$f(z) = \sum_{n\neq 0} A_f(n)\sqrt{y}K_{it_j}(2\pi|y|n)e^{2\pi inx}$$

if f is a Maass form with eigenvalue $\lambda_j = \frac{1}{4} + t_j^2$, or

$$f(z) = \sum_{n\geq 1} A_f(n) n^{\frac{k-1}{2}} e^{2\pi inz}$$

if f is a holomorphic cusp form. We write the coefficients of f as $a_f(n) = A_f(n)n^{\kappa(f)}$ with

$$\kappa(f) = \begin{cases} \frac{k-1}{2} & \text{if f is a holomorphic cusp form,} \\ 0 & \text{if f is a Maass form,} \end{cases}$$

Then $n^{\kappa(f)}$ conjecturally normalizes the coefficients $A_f(n)$ correctly depending on whether f is a holomorphic cusp form or a Maass form. It is expected that $A_f(n) \ll n^{\epsilon}$, but in general it is only known that

$$A_f(n) \ll n^{\alpha(f)+\epsilon}$$

where

$$\alpha(f) = \begin{cases} 0 & \text{if f is full-integral weight holomorphic,} \\ \frac{3}{16} & \text{if f is half-integral weight holomorphic,} \\ \frac{7}{64} & \text{if f is a Maass form.} \end{cases}$$

[†]The story behind the scope of this paper is a bit interesting, as it was written at the same time as [HKLDW17c]. Tom Hulse had learned of a set of criteria guaranteeing sign changes from a paper of Ram Murty, and he first thought of how to generalize the criteria to sequences of real and complex coefficients. I had thought it was possible to further generalize towards smoothed sums, but we failed in this regard. This morphed into our generalization, stating how to translate from analytic properties of Dirichlet series directly into sign-change results of the coefficients. We each focused on the parts that interested us most: I was interested in $S_f(n)$ and $S_f^{\nu}(n)$ (defined below), Hulse was interested in real and complex coefficients, and Ieong Kuan was interested in GL(3). In the end, each aspect strengthened the overall paper and led to a nice unified description of somewhat disjoint sign-change results.

Define the partial sums of normalized coefficients $S_f^\nu(n)$ as

$$S_f^\nu(n) = \sum_{m \le n} \frac{a_f(m)}{m^\nu}$$

Then by studying the Dirichlet series $D(s, S_f^\nu, \overline{S_g^\nu})$ and $D(s, S_f^\nu)$ in [HKLDW16], we proved the following theorem.

Theorem 4.1.4. *Let f be a weight 0 Maass form or holomorphic cusp form as described above. Suppose that $0 \le \nu < \kappa(f) + \frac{1}{6} - \frac{2\alpha(f)}{3}$. If there is a coefficient $a_f(n)$ such that $\operatorname{Re} a_f(n) \neq 0$ (resp. $\operatorname{Im} a_f(n) \neq 0$), then the sequence $\{\operatorname{Re} S_f^\nu(n)\}_{n\in\mathbb{N}}$ (resp. $\{\operatorname{Im} S_f^\nu(n)\}_{n\in\mathbb{N}}$) has at least one sign change for some $n \in [X, X + X^{r(\nu)}]$ for $X \gg 1$, where*

$$r(\nu) = \begin{cases} \frac{2}{3} + \frac{2\alpha(f)}{3} + \epsilon & \text{if } \nu < \kappa(f) + \frac{\alpha(f)}{3} - \frac{1}{6}, \\ \frac{2}{3} + \frac{2\alpha(f)}{3} + \Delta + \epsilon & \text{if } \nu = \kappa(f) + \frac{\alpha(f)}{3} - \frac{1}{6} + \Delta, 0 \le \Delta < \frac{1}{3} - \frac{2\alpha(f)}{3}. \end{cases}$$

In other words, we showed high regularity of the sign changes of sums of normalized coefficients, depending on the amount of normalization. As should be expected, higher amounts of normalization lead to fewer guaranteed sign changes.

However, we show that it is possible to take ν slightly larger than $\kappa(f)$, so that the individual coefficients $a_f(n)/n^\nu$ are each decaying in size. For example, for full-integer weight holomorphic cusp forms, we can take

$$\nu = \frac{k-1}{2} + \frac{1}{6} - \epsilon$$

and guarantee at least one sign change in $\{S_f^\nu(n)\}_{n\in\mathbb{N}}$ for some n in $[X, 2X]$ for sufficiently large X. Yet for this normalization, we have

$$S_f^\nu(n) = \sum_{m \le n} \frac{a_f(m)}{m^{\frac{k-1}{2} + \frac{1}{6} - \epsilon}},$$

so that the coefficients are *decaying* and look approximately like $n^{-1/6}$. It is a remarkable fact that the coefficients are arranged in such a way that there are still infinitely many sign regularly-spaced sign changes even though they are *over-normalized*.

This suggests a certain regularity of the sign changes of individual coefficients $a_f(n)$, but it is challenging to describe the exact nature of this regularity.

4.2 Directions for Further Investigation: Non-Cusp Forms

In the investigations carried out thus far, we have taken f to be a cusp form. But one can attempt to perform the same argument on sums of coefficients of noncuspidal automorphic forms.

One particular example would be to consider sums of the form

$$S_{\theta^k}(n) = \sum_{m \leq n} r_k(m),$$

where $r_k(m)$ is the number of ways of representing m as a sum of k squares. This is equivalent to the Gauss k-dimensional sphere problem, which asks how many integer lattice points are contained in $B_k(\sqrt{n})$, the k-dimensional sphere of radius $\sqrt{n}$ centered at the origin? A (very good) first approximation is that there are approximately $\mathrm{Vol}B_k(\sqrt{n})$ points within the sphere, so the question is really to understand the size of the discrepancy

$$P_k(n) := S_{\theta^k}(n) - \mathrm{Vol}B_k(\sqrt{n}).$$

My collaborators and I have been focusing our attention on this problem. In the recently submitted paper [HKLDW17b], we proved that $D(s, S_{\theta^k} \times S_{\theta^k})$ and $D(s, P_k \times P_k)$ have meromorphic continuation to the complex plane for $k \geq 3$. Using these continuations, we were able to prove a smooth estimate of a similar flavor as in Theorem 3.5.3.

Theorem 4.2.1. *For $k \geq 3$ and any $\epsilon > 0$,*

$$\sum_{n \geq 1} P_k(n)^2 e^{-n/X} = \delta_{[k=3]} C' X^2 \log X + C X^{k-1} + \delta_{[k=4]} C'' X^{\frac{5}{2}} + O(X^{k-2+\epsilon}),$$

where $\delta_{[k=n]}$ is 1 if $k = n$ and is 0 otherwise. Similarly,

$$\int_0^\infty P_k(t)^2 e^{-t/X} dt = \delta_{[k=3]} D' X^2 \log X + D X^{k-1} + \delta_{[k=4]} D'' X^{\frac{5}{2}} + O(X^{k-2+\epsilon}).$$

These two statements can be thought of as discrete and continuous Laplace transforms of the mean square error in the Gauss k-dimensional Sphere problem. We are also able to prove results concerning sharp sums and integrals.

Theorem 4.2.2. *For each $k \geq 3$, there exists $\lambda > 0$ such that*

$$\sum_{n \leq X} P_k(n)^2 = \delta_{[k=3]} C' X^{k-1} \log X + C X^{k-1} + O_\lambda(X^{k-1-\lambda}).$$

Similarly, we also have

$$\int_0^X (P_k(x))^2 dx = \delta_{[k=3]} D' X^{k-1} \log X + D X^{k-1} + O_\lambda(X^{k-1-\lambda}).$$

In the dimension 3 case, this is the first known polynomial savings on the error term, and represents the first major improvement over a result from Jarnik in 1940 [Jar40], which achieved only $\sqrt{\log X}$ savings.

In [HKLDW17b], we do not prove what λ is. In forthcoming work, we will consider the size of λ. We are also working on extending the techniques and results to the classical Gauss circle problem, when $k = 2$.

Remark 4.2.3. There are limitations to this technique. We can only consider forms f for which we understand the shifted convolution sum coming from $f \times f$ sufficiently well. So we are not capable of understanding sums of coefficients of Maass forms at this time, since shifted convolution sums of Maass forms with Maass forms remain mysterious.

Chapter Five

On Lattice Points on Hyperboloids

5.1 Introduction

A one-sheeted d-dimensional hyperboloid $\mathcal{H}_{d,h}$ is a surface satisfying the equation

$$X_1^2 + \cdots + X_{d-1}^2 = X_d^2 + h$$

for some fixed positive integer h. In this chapter, we investigate the number of integer points lying on the hyperboloid $\mathcal{H}_{d,h}$. In particular, we investigate the asymptotics for the number $N_{d,h}(R)$ of integer points $\boldsymbol{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d$ lying on $\mathcal{H}_{d,h}$ and within the ball $\|\boldsymbol{m}\|^2 \leq R$ for large R. Stated differently, if $B(\sqrt{R})$ is the ball of radius $\sqrt{R}$ in $\mathbb{R}^d$, centered at the origin, then

$$N_{d,h}(R) = \#\big(\mathbb{Z}^d \cap \mathcal{H}_{d,h} \cap B(\sqrt{R})\big).$$

Heuristically, one should expect to be capable of determining the leading term asymptotic using the circle method on hyperboloids $\mathcal{H}_{d,h}$ of sufficiently high dimension. More recently, Oh and Shah [OS14] used ergodic methods to study the three-dimensional hyperboloid $\mathcal{H}_{3,h}$ when h is a positive square. They proved the following theorem.

Theorem 5.1.1. *Oh and Shah Suppose that h is a square. On $\mathcal{H}_{3,h}$, as $X \to \infty$,*

$$N_{d,h}(X) = cX^{\frac{1}{2}} \log X + O\big(X^{\frac{1}{2}} (\log X)^{\frac{3}{4}}\big)$$

for some constant $c > 0$.

In this chapter, we sharpen and extend this theorem to any dimension $d \geq 3$ and any integral $h \geq 1$. We also prove a smoothed analogue, including smaller-order growth terms. The primary result is the following theorem.

Theorem 5.1.2. *Let $d \geq 3$ and $h \geq 1$ be integers. Let $N_{d,h}(R)$ denote the number of integer points $\boldsymbol{m}$ on the hyperboloid $\mathcal{H}_{d,h}$ with $\|\boldsymbol{m}\|^2 \leq R$. Then for any $\epsilon > 0$,*

$$N_{d,h}(R) = \delta_{[d=3]}\delta_{[h=a^2]} C_3' R^{\frac{1}{2}} \log R + C_d R^{\frac{d}{2}-1} + O(R^{\frac{d}{2}-1-\lambda(d)+\epsilon}).$$

Here the Kronecker δ expressions indicate that the first term only occurs if $d = 3$ and if h is a square, and $\lambda(d) > 0$ is a constant depending only on the dimension d.

This Theorem is presented in greater detail as Theorem 5.5.4, including the description of $\lambda(d)$. When $d = 3$, the power savings $\lambda(d)$ is exactly $\frac{1}{44}$. As d gets larger, $\lambda(d)$ grows and limits towards $\frac{1}{6}$. Note that there is an error term with polynomial savings, which is a significant improvement over previous results. As a corollary, one recovers the Theorem of Oh and Shah.

In addition to the sharp estimate of Theorem 5.1.2, we consider smoothed approximations to $N_{d,h}(R)$. In (1.1), we show that $N_{d,h}(R) = \sum_{2m^2+h\leq R} r_{d-1}(m^2+h)$. Then sums of the form

$$\sum_{m\in\mathbb{Z}} r_{d-1}(m^2+h)e^{-\frac{2m^2+h}{R}}$$

count the number of points $\boldsymbol{m}$ on $\mathcal{H}_{d,h}$ with exponential decay in $\|\boldsymbol{m}\|$ once $\|\boldsymbol{m}\|^2 \geq R$. This smoothed sum should be thought of as giving a smooth approximation to $N_{d,h}(R)$. Through the methodology of this chapter, we prove the following smooth estimate.

Theorem 5.1.3. *Let $d \geq 3$ and $h \geq 1$ be integers. Then for each h and d, there exist constants C' and C_m such that for any $\epsilon > 0$,*

$$\begin{aligned}\sum_{m\in\mathbb{Z}} & r_{d-1}(m^2+h)e^{-(2m^2+h)/X} \\ &= \delta_{[d=3]}\delta_{[h=a^2]}C'X^{\frac{1}{2}}\log X + \sum_{0\leq m<\lceil\frac{d}{2}-1\rceil} C_m X^{\frac{d}{2}-1-\frac{m}{2}} + O(X^{\frac{d}{4}-\frac{1}{2}+\epsilon}).\end{aligned}$$

Here, $\delta_{[condition]}$ is a Kronecker δ and evaluates to 1 if the condition is true and 0 otherwise.

See Theorem 5.5.1 in §5.5 for a more complete statement. This Theorem suggests that for dimensions greater than 4, there may be secondary main terms with lower power contributions.

Theorems 5.1.2 and 5.1.3 can be thought of as average order estimates of the function $r_{d-1}(m^2+h)$. In particular, for $2m^2+h \leq R$, the average value of $r_{d-1}(m^2+h)$ is about $R^{\frac{d-1}{2}-1}$. In the process of proving 5.1.2, we also prove that this average order estimate holds on short-intervals, i.e. intervals around R of length much less than R.

Theorem 5.1.4. *Let $k \geq \frac{1}{2}$ be a full or half-integer. Then for each dimension d, there is a constant $\lambda(d) > 0$ such that*

$$\sum_{|2m^2+h-X|<X^{1+\epsilon-\lambda(d)}} r_{d-1}(m^2+h) \ll X^{\frac{d}{2}-1+\epsilon-\lambda(d)}.$$

The constant $\lambda(d)$ is the same constant as in Theorem 5.1.2.

This Theorem can be roughly interpreted to count the number of lattice points $\boldsymbol{m}$ on $\mathcal{H}_{d,h}$ with $\|\boldsymbol{m}\|$ very near X, or equivalently counting the number of lattice points within a sphere of radius slightly larger than $\sqrt{X}$ and outside of a sphere of radius a slightly smaller than $\sqrt{X}$. This Theorem can be compared to the main theorem in short intervals in [HKLDW17c], as described in Chapter 4.

Remark 5.1.5. To make heuristic sense of Theorem 5.1.4, note that there are on the order of $X^{\frac{1}{2}+\epsilon-\lambda(d)}$ integers m such that $|2m^2+h-X| < X^{1+\epsilon-\lambda(d)}$. Therefore, if each of these values of $r_{d-1}(2m^2+h)$ is approximately the size we expect, $X^{\frac{d-1}{2}-1}$, then the total size should be

$$X^{\frac{d-1}{2}-1} \cdot X^{\frac{1}{2}+\epsilon-\lambda(d)} = X^{\frac{d}{2}-1+\epsilon-\lambda(d)},$$

which is exactly what is shown in Theorem 5.1.4.

Overview of Methodology

In order to count points on hyperboloids, let $d = 2k+2$ (where k may be a half-integer). Then in

$$X_1^2 + \cdots + X_{2k+1}^2 = X_{2k+2}^2 + h,$$

notice that for a point $\boldsymbol{X}$ on the hyperboloid,

$$(X_1^2 + \cdots + X_{2k+1}^2) + X_{2k+2}^2 \leq R \iff 2X_{2k+2}^2 + h \leq R.$$

It suffices to consider those points on the hyperboloid with $2X_{2k+2}^2 + h \leq R$. Recall the notation that $r_d(n)$ is the number of representations of n as a sum of d squares. Then, breaking the hyperboloid into each possible value of $X_{2k+2}^2 + h$ and summing across the number of representations as sums of squares, we have that

$$N_{d,h}(R) = \sum_{2X_{2k+2}^2+h\leq R} r_{2k+1}(X_{2k+2}^2+h) = \sum_{2m^2+h\leq R} r_{2k+1}(m^2+h). \tag{1.1}$$

We will find the number of points on the hyperboloid by estimating this last sum.

Consider the automorphic function

$$\widetilde{V}(z) = \theta^{2k+1}(z)\overline{\theta(z)}y^{\frac{k+1}{2}},$$

where

$$\theta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z}$$

is the classical Jacobi theta function. Heuristically, the hth Fourier coefficient of $\widetilde{V}(z)$ is a weighted version of the sum $\sum_m r_{2k+1}(m^2+h)$, and so proper analysis of the the hth Fourier coefficient of $\widetilde{V}(z)$ will give an estimate for $N_{d,h}(R)$.

More completely, let $P_h(z,s)$ denote a Poincaré series that isolates the hth Fourier coefficient. Then we will have that

$$\frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})}\langle P_h(\cdot,s), \widetilde{V}\rangle = \sum_{m\in\mathbb{Z}} \frac{r_{2k+1}(m^2+h)}{(2m^2+h)^{s+\frac{k-1}{2}}}.$$

In order to understand the meromorphic properties of this Dirichlet series, we will use a spectral expansion of the Poincaré series and understand each of the terms in the spectral decomposition. As $\widetilde{V}(z)$ is not square integrable, it is necessary to modify $\widetilde{V}(z)$ by cancelling out the growth. We do that in the next section by subtracting carefully chosen Eisenstein series.

Once the meromorphic properties of this Dirichlet series are understood, it only remains to perform some classical cutoff integral transforms. In Section 5.5, we apply three Mellin integral transforms described in Chapter 2 and perform classical integral analysis in order to prove our main theorems of this chapter.

5.2 Altering $\widetilde{V}$ to be Square-Integrable

From the transformation laws of $\theta(z)$, we see that $\widetilde{V}$ satisfies the transformation law

$$\widetilde{V}(\gamma z) = \frac{\varepsilon_d^{-2k}\left(\frac{c}{d}\right)^{2k}(cz+d)^k}{|cz+d|^k}\widetilde{V}(z) \tag{2.1}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, and where

$$\varepsilon_d = \begin{cases} 1 & d \equiv 1 \pmod 4 \\ i & d \equiv 3 \pmod 4 \end{cases}$$

is the sign of the dth Gauss sum. Therefore when k is an integer, $\widetilde{V}$ is a modular form of full-integral weight k of nebentypus $\chi(\cdot) = \left(\frac{-1}{\cdot}\right)^k$ on $\Gamma_0(4)$. When k is a half-integer, $\widetilde{V}$ is a modular form of half-integral weight k on $\Gamma_0(4)$ with a normalized theta multiplier system as described in (2.1).

Under the action of $\Gamma_0(4)$, the quotient $\Gamma_0(4)\backslash\mathcal{H}$ has three cusps: at 0, $\frac{1}{2}$, and ∞. We use $E_{\mathfrak{a}}^k(z,w)$ to denote the Eisenstein series of weight k associated to the cusp $\mathfrak{a}$, as detailed extensively in §2.2. We will soon see that $\widetilde{V}$ is non-cuspidal, and we will analyze the behavior of $\widetilde{V}$ at each of the cusps. In doing so, we will prove the following.

Proposition 5.2.1. *For $k \geq 1$, define $V(z)$ as*

$$V(z) := \widetilde{V}(z) - E_\infty^k(z, \tfrac{k+1}{2}) - E_0^k(z, \tfrac{k+1}{2}).$$

Then $V(z)$ is in $L^2(\Gamma_0(4)\backslash\mathcal{H}, k)$.

In the case when $k = \frac{1}{2}$, we define

$$V(z) := \widetilde{V}(z) - \operatorname*{const}_{w=\frac{3}{4}} E_\infty^k(z,w) - \operatorname*{const}_{w=\frac{3}{4}} E_0^k(z,w),$$

where $\mathrm{const}_{w=c} f(w)$ *refers to the constant term in the Laurent expansion of* $f(w)$ *expanded at* $w = c$*. Then* $V(z)$ *is in* $L^2(\Gamma_0(4)\backslash\mathcal{H}, \frac{1}{2})$*.*

Proof. Writing $\widetilde{V}$ directly as

$$\widetilde{V}(z) = \sum_{m_1,\dots,m_{2k+2}\in\mathbb{Z}} y^{\frac{k+1}{2}} e^{2\pi i x(m_1^2+\cdots m_{2k+1}^2 - m_{2k+2}^2)} e^{-2\pi y(m_1^2+\cdots+m_{2k+2}^2)}$$

shows that all terms have significant exponential decay in y, except when $m_1 = \cdots = m_{2k+2} = 0$, in which case there is the term $y^{\frac{k+1}{2}}$. Correspondingly, at the ∞ cusp, $\widetilde{V}(z)$ grows like $y^{\frac{k+1}{2}}$. However the function $\widetilde{V}(z) - y^{\frac{k+1}{2}}$ has exponential decay as $y \to \infty$.

At the 0 cusp, we use $\sigma_0 = \left(\begin{smallmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{smallmatrix}\right)$, a matrix in $\mathrm{SL}(2,\mathbb{R})$ taking 0 to ∞, and directly compute

$$\widetilde{V}\big|_{\sigma_0}(z) = \theta^{2k+1}\Big(\frac{-1}{4z}\Big)\overline{\theta\Big(\frac{-1}{4z}\Big)}\,\mathrm{Im}^{\frac{k+1}{2}}\Big(\frac{-1}{4z}\Big)\frac{|-2iz|^k}{(-2iz)^k}$$

At the $\frac{1}{2}$ cusp, $\widetilde{V}$ has exponential decay because each $\theta(z)$ factor has exponential decay there.

Thus $\widetilde{V}$ grows like $y^{\frac{k+1}{2}}$ at the ∞ and 0 cusps, and has exponential decay at the $\frac{1}{2}$ cusp. To cancel and better understand these growth terms, we subtract spectral Eisenstein series associated to the cusps 0 and ∞ with spectral parameter chosen so that the leading growth of the Eisenstein series perfectly cancels the growth of $\widetilde{V}$. We will use the properties of the full and half-integral weight Eisenstein series associated to the cusp $\mathfrak{a}$, $E_{\mathfrak{a}}^k(z,w)$, as described more fully in §2.2. In particular, it is shown in §2.2 that the constant terms in the Fourier series of the Eisenstein series $E_{\mathfrak{a}}^k(z,w)$, expanded at the cusp $\mathfrak{a}$, is of the shape

$$y^w + c(w)y^{1-w}$$

for a constant $c(w)$ depending on w. Therefore, specializing the parameter $w = \frac{k+1}{2}$, the leading term from the constant term of each Eisenstein series perfectly cancels the growth of $\widetilde{V}$ at each cusp. Further, each Eisenstein series is small at each cusp other than its associated cusp, so for instance $E_\infty^k(z, \frac{k+1}{2})$ cancels the $y^{\frac{k+1}{2}}$ at the ∞ cusp and is otherwise small at each other cusp (see [Iwa97] for more).

However, when k is half-integral weight, the Eisenstein series $E_{\mathfrak{a}}^k(z,w)$ has a pole at $w = \frac{3}{4}$. When $k = \frac{1}{2}$, corresponding to the dimension 3 hyperboloid, the two Eisenstein series $E_\infty^{\frac{1}{2}}(z,w)$ and $E_0^{\frac{1}{2}}(z,w)$ each have poles at $w = \frac{k+1}{2} = \frac{3}{4}$, and so we

cannot subtract them from $\widetilde{V}$ directly. Referring again to §2.2, it is clear that the constant term of the Laurent expansion at $w = \frac{3}{4}$ of each Eisenstein series contains the leading growth terms $y^{\frac{3}{4}}$. Since the constant term in the Laurent expansion is also modular, we conclude the $k = \frac{1}{2}$ case. □

5.3 Analytic Behavior

Let $P_h^k(z,s)$ denote the weight k Poincaré series

$$P_h^k(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \mathrm{Im}(\gamma z)^s e^{2\pi i h \gamma z} J(\gamma, z)^{-2k}$$

where

$$J(\gamma, z) = \frac{j(\gamma, z)}{|j(\gamma, z)|}$$

and $j(\gamma, z) = \theta(\gamma z)/\theta(z) = \varepsilon^{-1} \left(\frac{c}{d}\right) (cz + d)^{\frac{1}{2}}$, exactly as for the Eisenstein series defined in Chapter 2.

Our basic strategy is to understand the Petersson inner product $\langle P_h^k(\cdot, s), V(z) \rangle$ in two different ways. On the one hand, we will compute it directly, giving a Dirichlet series $D_h^k(s)$ with coefficients $r_{2k+1}(m^2 + h)$. On the other hand, we will take a spectral expansion of P_h^k and understand the meromorphic properties of each part of the spectral expansion.

5.3.1 Direct Expansion

We first understand $\langle P_h^k(\cdot, s), V \rangle$ directly, using the method of unfolding:

$$\begin{aligned} \langle P_h^k(\cdot, s), V \rangle &= \int \int_{\Gamma_0(4) \backslash \mathcal{H}} P_h(z, s) \overline{V(z)} \frac{dx dy}{y^2} \\ &= \int_0^\infty \int_0^1 y^{s-1} e^{2\pi i h z} \overline{V(z)} dx \frac{dy}{y}. \end{aligned}$$

Initially, we consider the case when $k > \frac{1}{2}$. We'll consider the three dimensional case, when $k = \frac{1}{2}$, afterwards.

Dimension ≥ 4

Writing $V = \widetilde{V} - E_\infty^k(z, \frac{k+1}{2}) - E_0^k(z, \frac{k+1}{2})$, we compute

$$\begin{aligned}&\langle P_h^k(\cdot, s), E_\infty^k(z, \tfrac{k+1}{2}) + E_0^k(z, \tfrac{k+1}{2})\rangle \\ &\quad = \frac{\overline{\rho_\infty^k(h, \frac{k+1}{2}) + \rho_0^k(h, \frac{k+1}{2})}}{(4\pi h)^{s-1}} \frac{\Gamma(s + \frac{k}{2} - \frac{1}{2})\Gamma(s - \frac{k}{2} - \frac{1}{2})}{\Gamma(s - \frac{k}{2})}.\end{aligned}$$

Expanding $\widetilde{V}$, we can compute the remaining x integral as

$$\begin{aligned}\int_0^1 \overline{\widetilde{V}(z)} e^{2\pi i h x} dx &= \int_0^1 \overline{\theta^{2k+1}(z)}\theta(z) y^{\frac{k+1}{2}} e^{2\pi i h x} dx \\ &= \sum_{m_1, \ldots, m_{2k+2}} y^{\frac{k+1}{2}} e^{2\pi y(m_1^2 + \cdots + m_{2k+2}^2)} \int_0^1 e^{-2\pi i x(m_1^2 + \cdots + m_{2k+1}^2 - m_{2k+2}^2 - h)} dx \\ &= \quad y^{\frac{k+1}{2}} \sum_{\substack{\boldsymbol{m} \in \mathbb{Z}^{2k+2} \\ m_1^2 + \cdots + m_{2k+1}^2 = m_{2k+2}^2 + h}} e^{-2\pi y(m_1^2 + \cdots m_{2k+2}^2)} = \quad y^{\frac{k+1}{2}} \sum_{\substack{\boldsymbol{m} \in \mathbb{Z}^{2k+2} \\ m_1^2 + \cdots + m_{2k+1}^2 = m_{2k+2}^2 + h}} e^{-2\pi y(2m_{2k+2}^2 + h)} \\ &= y^{\frac{k+1}{2}} \sum_{m \in \mathbb{Z}} r_{2k+1}(m^2 + h) e^{-2\pi y(2m^2 + h)}.\end{aligned}$$

To go from the penultimate line to the last line, we write $m = m_{2k+2}$ and count the number of representations of $m^2 + h$. We compute the remaining y integral

$$\int_0^\infty y^{s + \frac{k-1}{2}} \sum_{m \in \mathbb{Z}} r_{2k+1}(m^2 + h) e^{-2\pi y(2m^2 + h)} \frac{dy}{y} = \sum_{m \in \mathbb{Z}} \frac{r_{2k+1}(m^2 + h)}{(2m^2 + h)^{s + \frac{k-1}{2}}} \frac{\Gamma(s + \frac{k-1}{2})}{(2\pi)^{s + \frac{k-1}{2}}}.$$

Define

$$D_h^k(s) := \frac{(2\pi)^{s + \frac{k-1}{2}}}{\Gamma(s + \frac{k-1}{2})} \langle P_h^k(\cdot, s), V\rangle. \tag{3.1}$$

Then our computation above shows that for $k \geq 1$, $D_h^k(s)$ can be written as

$$\sum_{m \in \mathbb{Z}} \frac{r_{2k+1}(m^2 + h)}{(2m^2 + h)^{s + \frac{k-1}{2}}} - \mathfrak{E}_h^k(s)$$

where

$$\mathfrak{E}_h^k(s) = \frac{(2\pi)^{\frac{k+1}{2}} \overline{(\rho_\infty^k(h, \frac{k+1}{2}) + \rho_0^k(h, \frac{k+1}{2}))}}{(2h)^{s-1}} \frac{\Gamma(s - \frac{k}{2} - \frac{1}{2})}{\Gamma(s - \frac{k}{2})}.$$

Notice that $\mathfrak{E}_h^k(s)$ has poles at $s = \frac{k+1}{2} - m$ for $m \in \mathbb{Z}_{\geq 0}$, coming from the Gamma function in the numerator, and clear meromorphic continuation.[†]

[†] $\mathfrak{E}$ is an E in an old German font, which many mathematicians would pronounce as "fraktur E" or "mathfrak E." We use it because those terms come from Eisenstein series.

After applying Stirling's approximation to estimate the Gamma functions, we have the following proposition.

Proposition 5.3.1. *With the notation above and with $k \geq 1$, we have*

$$D_h^k(s) := \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})}\langle P_h^k(\cdot,s), V\rangle = \sum_{m\in\mathbb{Z}} \frac{r_{2k+1}(m^2+h)}{(2m^2+h)^{s+\frac{k-1}{2}}} - \mathfrak{E}_h^k(s).$$

The function $\mathfrak{E}_h^k(s)$ is analytic for $\operatorname{Re} s > \frac{1}{2}$ except for simple poles at $s = \frac{k+1}{2} - m$ for $m \in \mathbb{Z}_{\geq 0}$, and has meromorphic continuation to the plane. For s away from poles with $\operatorname{Re} s > \frac{1}{2}$, we have the bound

$$\mathfrak{E}_h^k(s) \ll (1+|s|)^{-\frac{1}{2}}.$$

Dimension 3

We proceed analogously, and write $V = \widetilde{V} - \operatorname{const}_{w=\frac{3}{4}} E_\infty^{\frac{1}{2}}(z,w) - \operatorname{const}_{w=\frac{3}{4}} E_0^{\frac{1}{2}}(z,w)$, initially for $\operatorname{Re} s \gg 1$. We now need to compute

$$\langle P_h^{\frac{1}{2}}(\cdot,s), \operatorname*{const}_{w=\frac{3}{4}} E_{\mathfrak{a}}^{\frac{1}{2}}(\cdot,w)\rangle = \operatorname*{const}_{w=\frac{3}{4}} \langle P_h^{\frac{1}{2}}(\cdot,s), E_{\mathfrak{a}}^{\frac{1}{2}}(\cdot,w)\rangle.$$

Computing the expression for the inner product $\langle P_h^{\frac{1}{2}}, E_{\mathfrak{a}}^{\frac{1}{2}}\rangle$ for $\operatorname{Re} s \gg 1$ directly and then taking the constant term in w gives that

$$\operatorname*{const}_{w=\frac{3}{4}} \langle P_h^{\frac{1}{2}}(\cdot,s), E_{\mathfrak{a}}^{\frac{1}{2}}(\cdot,w)\rangle = \operatorname*{const}_{w=\frac{3}{4}} \frac{\overline{\rho_{\mathfrak{a}}^{\frac{1}{2}}(h,w)}}{(4\pi h)^{s-1}} \frac{\Gamma(s+\overline{w}-1)\Gamma(s-\overline{w})}{\Gamma(s-\frac{1}{4})}. \tag{3.2}$$

We must now make sense of this constant term.

In §2.2, it is shown that $\rho_{\mathfrak{a}}^{\frac{1}{2}}(h,w)$ has a simple pole at $w = \frac{3}{4}$ if and only if h is a positive square, and otherwise is analytic. Thus the constant term (3.2) manifests in three ways:

(i) The constant terms of $\rho_{\mathfrak{a}}^{\frac{1}{2}}(h,w)$, $\Gamma(s+\overline{w}-1)$, and $\Gamma(s-\overline{w})$

(ii) The residue term of $\rho_{\mathfrak{a}}^{\frac{1}{2}}(h,w)$, the constant term of $\Gamma(s+\overline{w}-1)$, and the linear term of $\Gamma(s-\overline{w})$

(iii) The residue term of $\rho_{\mathfrak{a}}^{\frac{1}{2}}(h,w)$, the linear term of $\Gamma(s+\overline{w}-1)$, and the constant term of $\Gamma(s-\overline{w})$.

Together, these mean that (3.2) can be written as

$$\begin{aligned}
&\operatorname*{const}_{w=\frac{3}{4}} \langle P_h^{\frac{1}{2}}(\cdot,s), E_{\mathfrak{a}}^{\frac{1}{2}}(\cdot,w)\rangle = \\
&= \frac{\operatorname{const}_{w=\frac{3}{4}} \rho_{\mathfrak{a}}^{\frac{1}{2}}(h,w)\Gamma(s-\frac{1}{4})\Gamma(s-\frac{3}{4})}{(4\pi h)^{s-1}\Gamma(s-\frac{1}{4})} \\
&\quad + \frac{\operatorname{Res}_{w=\frac{3}{4}} \rho_{\mathfrak{a}}^{\frac{1}{2}}(h,w)}{(4\pi h)^{s-1}}\left(\frac{\Gamma'(s-\frac{1}{4})\Gamma(s-\frac{3}{4})}{\Gamma(s-\frac{1}{4})} + \frac{\Gamma'(s-\frac{3}{4})\Gamma(s-\frac{1}{4})}{\Gamma(s-\frac{1}{4})}\right) \qquad (3.3)\\
&= \frac{\operatorname{const}_{w=\frac{3}{4}} \rho_{\mathfrak{a}}^{\frac{1}{2}}(h,w)\Gamma(s-\frac{3}{4})}{(4\pi h)^{s-1}} \\
&\quad + \frac{\operatorname{Res}_{w=\frac{3}{4}} \rho_{\mathfrak{a}}^{\frac{1}{2}}(h,w)}{(4\pi h)^{s-1}}\left(\frac{\Gamma'(s-\frac{1}{4})\Gamma(s-\frac{3}{4})}{\Gamma(s-\frac{1}{4})} + \Gamma'(s-\tfrac{3}{4})\right).
\end{aligned}$$

This expression has clear meromorphic continuation to the plane, and the poles and analytic behavior can be determined from the individual Gamma functions. This expression has a simple pole at $s=\frac{3}{4}$, and when $\rho_{\mathfrak{a}}^{\frac{1}{2}}(s,w)$ has a pole at $w=\frac{3}{4}$, this expression has a double pole in s at $s=\frac{3}{4}$ coming from $\Gamma'(s-\frac{3}{4})$. These are the only poles in this expression when $\operatorname{Re} s > \frac{1}{2}$.

As in the case when $k \geq 1$, we define

$$D_h^{\frac{1}{2}}(s) := \frac{(2\pi)^{s-\frac{1}{4}}}{\Gamma(s-\frac{1}{4})}\langle P_h^{\frac{1}{2}}(\cdot,s), V\rangle. \qquad (3.4)$$

At each cusp $\mathfrak{a}$, we also define

$$\mathfrak{E}_{h,\mathfrak{a}}^{\frac{1}{2}}(s) := \frac{(2\pi)^{s-\frac{1}{4}}}{\Gamma(s-\frac{1}{4})} \operatorname*{const}_{w=\frac{3}{4}} \langle P_h^{\frac{1}{2}}(\cdot,s), E_{\mathfrak{a}}^{\frac{1}{2}}(\cdot,w)\rangle.$$

Notice that this is $(2\pi)^{s-\frac{1}{4}}\Gamma(s-\frac{1}{4})^{-1}$ times the expression in (3.3). Finally, define

$$\mathfrak{E}_h^{\frac{1}{2}}(s) := \mathfrak{E}_{h,\infty}^{\frac{1}{2}}(s) + \mathfrak{E}_{h,0}^{\frac{1}{2}}(s).$$

Then when $k \geq 1$, we have that

$$D_h^{\frac{1}{2}}(s) = \frac{(2\pi)^{s-\frac{1}{4}}}{\Gamma(s-\frac{1}{4})}\langle P_h^{\frac{1}{2}}(\cdot,s), V\rangle = \sum_{m\in\mathbb{Z}} \frac{r_3(m^2+h)}{(2m^2+h)^{s-\frac{1}{4}}} - \mathfrak{E}_h^{\frac{1}{2}}(s).$$

Although the intermediate steps are different, this final notation agrees with the notation for $k \geq 1$.

To roughly understand the growth of $\Gamma'(s)$, it suffices to use Cauchy's Integral Formula by examining (for $|\operatorname{Im} s| \gg 1$)

$$\Gamma'(s) = \frac{1}{2\pi i}\int_{\mathcal{B}_1(s)} \frac{\Gamma(z)}{(z-s)^2}dz \ll \max_{0\leq\theta\leq 2\pi} |\Gamma(s+e^{i\theta})|,$$

where $\mathcal{B}_1(s)$ is the circle of radius 1 around s. It is then straightforward to get rough bounds on $\Gamma'(s)$ through Stirling's Approximation.

Remark 5.3.2. It is possible to get much stronger bounds, but it will turn out that this rough bound suffices.

We gather the relevant details from this section into the following proposition.

Proposition 5.3.3. *With the notation above,*

$$D_h^{\frac{1}{2}}(s) = \frac{(2\pi)^{s-\frac{1}{4}}}{\Gamma(s-\frac{1}{4})}\langle P_h^{\frac{1}{2}}(\cdot, s), V\rangle = \sum_{m\in\mathbb{Z}} \frac{r_3(m^2+h)}{(2m^2+h)^{s-\frac{1}{4}}} - \mathfrak{E}_h^{\frac{1}{2}}(s).$$

Further, $\mathfrak{E}_h^{\frac{1}{2}}(s)$ has meromorphic continuation to the plane, and is analytic for $\operatorname{Re} s > \frac{1}{2}$ except for a pole at $s = \frac{3}{4}$. If h is a square, this is a double pole. If h is not a square, then this is a simple pole.

For s away from the pole at $\frac{3}{4}$, $\operatorname{Re} s > \frac{1}{2}$, we have the bound

$$\mathfrak{E}_h^{\frac{1}{2}}(s) \ll (1+|s|)^{\frac{1}{2}}.$$

5.3.2 Spectral Expansion

Fix an orthonormal basis of Maass forms of weight k for $L^2(\Gamma_0(4)\backslash\mathcal{H}, k)$. This basis consists of Maass forms $\{\mu_j(z)\}$ of types $\frac{1}{2}+it_j$ and corresponding eigenvalues $\frac{1}{4}+t_j^2$, each with expansion

$$\mu_j(z) = \sum_{n\neq 0} \rho_j(n) W_{\frac{n}{|n|}\frac{k}{2}, it_j}(4\pi|n|y)e^{2\pi inx},$$

as well as a finite number of Maass forms $\mu_{j,\ell}(z)$ with eigenvalues $\frac{\ell}{2}(1-\frac{\ell}{2})$ with $1 \leq \ell \leq k$ for ℓ a (possibly half-integer) satisfying $\ell \equiv k \bmod 2$, each with expansion

$$\mu_{j,\ell}(z) = \sum_{n\neq 0} \rho_{j,\ell}(n) W_{\frac{n}{|n|}\frac{k}{2}, \frac{\ell-1}{2}}(4\pi|n|y)e^{2\pi inx}.$$

Note that for each ℓ, there are finitely many such Maass forms, as these come from holomorphic cusp forms of weight ℓ. These Maass forms contribute the so-called *bottom of the spectrum*, as described in [GH11, Chapter 3].

Then P_h^k has a Selberg Spectral decomposition [IK04, Gol06] of the form

$$P_h^k(z,s) = \sum_j \langle P_h^k(\cdot, s), \mu_j\rangle \mu_j(z) + \sum_{\frac{1}{2}\leq\ell\leq k} \sum_j \langle P_h^k(\cdot, s), \mu_{j,\ell}\rangle \mu_{j,\ell}(z) \tag{3.5}$$

$$+\sum_{\mathfrak{a}}\langle P_h^k(\cdot,s),R_{\mathfrak{a}}^k\rangle R_{\mathfrak{a}}^k(z) \tag{3.6}$$

$$+\frac{1}{4\pi i}\sum_{\mathfrak{a}}\int_{(\frac{1}{2})}\langle P_h^k(\cdot,s),E_{\mathfrak{a}}^k(\cdot,u)\rangle E_{\mathfrak{a}}^k(z,u)du. \tag{3.7}$$

In this expansion, line (3.5) is the *discrete part of the spectrum*, line (3.6) is the *residual part of the spectrum*, and line (3.7) is the *continuous part of the spectrum*. The sums over $\mathfrak{a}$ are sums over the three cusps of $\Gamma_0(4)$. Note that the residual part of the spectrum exists only when k is a half-integer, in which case

$$R_{\mathfrak{a}}^k(z)=\operatorname*{Res}_{w=\frac{3}{4}}E_{\mathfrak{a}}^k(z,w). \tag{3.8}$$

Each of the inner products against P_h^k can be directly evaluated. These computations are very similar to those computations in §2.2 and §5.3.1, and we omit them. We collect these together in the following lemma.

Lemma 5.3.4. *Maintaining the notation above, we have*

$$\begin{aligned}
\langle P_h^k(\cdot,s),\mu_j\rangle &= \frac{\overline{\rho_j(h)}}{(4\pi h)^{s-1}}\frac{\Gamma(s-\frac{1}{2}+it_j)\Gamma(s-\frac{1}{2}-it_j)}{\Gamma(s-\frac{k}{2})}\\
\langle P_h^k(\cdot,s),\mu_{j,\ell}\rangle &= \frac{\overline{\rho_{j,\ell}(h)}}{(4\pi h)^{s-1}}\frac{\Gamma(s+\frac{\ell}{2}-1)\Gamma(s-\frac{\ell}{2})}{\Gamma(s-\frac{k}{2})}\\
\langle P_h(\cdot,s),R_{\mathfrak{a}}^k\rangle &= \frac{\operatorname{Res}_{w=\frac{3}{4}}\overline{\rho_{\mathfrak{a}}^k(h,w)}}{(4\pi h)^{s-1}}\frac{\Gamma(s-\frac{1}{4})\Gamma(s-\frac{3}{4})}{\Gamma(s-\frac{k}{2})}\\
\langle P_h(\cdot,s),E_{\mathfrak{a}}^k(\cdot,\tfrac{1}{2}+it)\rangle &= \frac{\overline{\rho_{\mathfrak{a}}(h,\frac{1}{2}+it)}}{(4\pi h)^{s-1}}\frac{\Gamma(s-\frac{1}{2}+it)\Gamma(s-\frac{1}{2}-it)}{\Gamma(s-\frac{k}{2})}.
\end{aligned}$$

Here, $\rho_j(h)$ is the hth coefficient of μ_j, $\rho_{j,\ell}(h)$ is the hth coefficient of $\mu_{j,\ell}$, $\rho_{\mathfrak{a}}(h,\frac{1}{2}+it)$ is the hth coefficient of $E_{\mathfrak{a}}^k(z,\frac{1}{2}+it)$, and $R_{\mathfrak{a}}^k$ is as in (3.8).

5.3.3 Meromorphic Continuation of $\langle P_h^k, V\rangle$

In order to provide a meromorphic continuation for $D_h^k(s)$ (defined in (3.1) and (3.4)), we provide a meromorphic continuation for the expression coming from the spectral decomposition of $P_h^k(z,s)$. Inserting the spectral decomposition of $P_h^k(z,s)$ into $\langle P_h^k(\cdot,s),V\rangle$, we get

$$\langle P_h^k(\cdot,s),V\rangle=\sum_j\langle P_h^k(\cdot,s),\mu_j\rangle\langle\mu_j,V\rangle+\sum_{\frac{1}{2}\le\ell\le k}\sum_j\langle P_h^k(\cdot,s),\mu_{j,\ell}\rangle\langle\mu_{j,\ell},V\rangle \tag{3.9}$$

$$+\sum_{\mathfrak{a}}\langle P_h^k(\cdot,s),R_{\mathfrak{a}}^k\rangle\langle R_{\mathfrak{a}}^k,V\rangle \tag{3.10}$$

$$+\frac{1}{4\pi i}\sum_{\mathfrak{a}}\int_{(\frac{1}{2})}\langle P_h^k(\cdot,s),E_{\mathfrak{a}}^k(\cdot,u)\rangle\langle E_{\mathfrak{a}}^k(\cdot,u),V\rangle du, \tag{3.11}$$

where we have again separated the expressions into separate lines for the discrete spectrum, the residual spectrum, and the continuous spectrum. To study the meromorphic continuation, we provide separate meromorphic continuations for the discrete spectrum, the residual spectrum, and the continuous spectrum in turn.

Discrete Spectrum

Consider the discrete spectrum appearing in line (3.9), which we rewrite for convenience:

$$\langle P_h^k(\cdot,s),V\rangle=\sum_j\langle P_h^k(\cdot,s),\mu_j\rangle\langle\mu_j,V\rangle+\sum_{\frac{1}{2}\le\ell\le k}\sum_j\langle P_h^k(\cdot,s),\mu_{j,\ell}\rangle\langle\mu_{j,\ell},V\rangle.$$

Analysis of the discrete spectrum naturally breaks into two categories: analysis of the finitely many Maass forms $\{\mu_{j,\ell}\}_{j,\ell}$ coming from holomorphic cusp forms of weight ℓ, and analysis of the infinitely many Maass forms $\{\mu_j\}_j$ with corresponding types $\frac{1}{2}=it_j$. Taking inspiration from [GH11, §3.10], we refer to the Maass forms $\{\mu_{j,\ell}\}_{j,\ell}$ as the *old discrete spectrum*, and the Maass forms $\{\mu_j\}_j$ as the *new discrete spectrum*. We will prove the following proposition in this section.

Proposition 5.3.5. *Write $s=\sigma+it$. The discrete spectrum component has meromorphic continuation to the plane, and*

(i) For $\operatorname{Re}s>\frac{1}{2}$, the new discrete spectrum is analytic and satisfies the bound

$$\sum_j\langle P_h^k(\cdot,s),\mu_j\rangle\langle\mu_j,V\rangle\ll_{h,\epsilon}(1+|t|)^{\frac{7k}{2}+\frac{15}{2}+\epsilon}e^{-\frac{\pi}{2}|t|}.\qquad(\tfrac{1}{2}<\operatorname{Re}s<1)$$

The new discrete spectrum has a line of poles on $\operatorname{Re}s=\frac{1}{2}$.

(ii) For $\operatorname{Re}s>\frac{k}{2}-1$, the old discrete spectrum is analytic and satisfies the bound

$$\sum_{\frac{1}{2}\le\ell\le k}\sum_j\langle P_h^k(\cdot,s),\mu_{j,\ell}\rangle\langle\mu_{j,\ell},V\rangle\ll(1+|t|)^{\frac{k}{2}-\frac{3}{2}+\sigma}e^{-\frac{\pi}{2}|t|}.$$

In the region $\operatorname{Re}s>0$, the old discrete spectrum has simple poles at $s=\frac{k}{2}-1-m$ for $m\in\mathbb{Z}_{\ge0}$.

In order to prove this proposition, we prove a sequence of lemmata. We first bound the infinite sum in the new discrete spectrum.

Lemma 5.3.6. *With μ_j coming from the new discrete spectrum, and with the same notation as above, we have*

$$\sum_{T\leq|t_j|\leq 2T}\rho_j(h)\langle\mu_j,V\rangle=\sum_{T\leq|t_j|\leq 2T}\rho_j(h)\langle\mu_j,\theta^{2k+1}\overline{\theta}y^{\frac{k+1}{2}}\rangle\ll T^{3k+8+\epsilon}.$$

Proof. First note that $E_{\mathfrak{a}}^k(z,\frac{k+1}{2})$ is orthogonal to the cusp form μ_j, which gives the first equality. We recognize $\theta(z)y^{1/4}$ as a constant times $\operatorname{Res}_{u=\frac{3}{4}}E_\infty^{\frac{1}{2}}(z,u)$. Performing this on $\overline{\theta}$ transforms each inner product into

$$\operatorname*{Res}_{u=\frac{3}{4}}\langle\mu_j,\theta^{2k+1}\overline{E_\infty^{\frac{1}{2}}(\cdot,u)y^{\frac{2k+1}{4}}}\rangle.$$

Using the standard unfolding argument on the Eisenstein series (which uses similar methodology to the computations of $\langle P,V\rangle$ in §5.3.1), we see that this is equal to

$$\operatorname*{Res}_{u=\frac{3}{4}}\sum_{n\geq 1}\frac{r_{2k+1}(n)\rho_j(n)}{(4\pi n)^{u+\frac{k}{2}-\frac{3}{4}}}\frac{\Gamma(u+\frac{k}{2}-\frac{3}{4}+\frac{1}{2}+it_j)\Gamma(u+\frac{k}{2}-\frac{3}{4}+\frac{1}{2}-it_j)}{\Gamma(u+\frac{1}{4})}.$$

We bound the size of this residue by first proving a bound in u using the Phragmén-Lindelöf principle, and then using Cauchy's Residue Theorem to bound the sum.

From the average estimate $r_d(n)\approx n^{\frac{d}{2}-1}$, one can show that the summation converges trivially absolutely for $\operatorname{Re}u>\frac{k}{2}+\frac{5}{4}$. Applying Phragmé-Lindelöf and using Stirling's approximation then shows that

$$\rho_j(h)\langle\mu_j,\theta^{2k+1}\overline{\theta}y^{\frac{k+1}{2}}\rangle\ll\rho_j(h)(1+|t_j|)^{3k+6+\epsilon}e^{-\pi|t_j|}.$$

Remark 5.3.7. Note that as a residue in u is being taken, the relevant bound from the t_j contribution, which is entirely determined by the factors

$$\rho_j(1)\Gamma(u+\tfrac{k}{2}-\tfrac{1}{4}+it_j)\Gamma(u+\tfrac{k}{2}-\tfrac{1}{4}-it_j).$$

Heuristically, the final bound can be attained just from using the Phragmén-Lindelöf Convexity principle on this piece.

As noted in [HKKrL16], it is possible to understand bound $\rho_j(h)$ on average over j. Through the standard Kuznetsof Trace Formula and arguing as in [HHR13, Section 4], one can show that $\rho_j(h)\ll_{\epsilon,h}e^{\frac{\pi}{2}|t_j|}(1+|t_j|)^\epsilon$ (on average over j) for full integral weight k. By using Proskurin's generalization of the Kuznetsof Trace Formula, described in [Duk88], one can show the same for half integral weight k. (This is closely related to an argument in [HKKrL16]).

Remark 5.3.8. For full-integral weight Maass forms, Goldfeld, Hoffstein, and Lieman [GHL94] showed that $\rho_j(h) \ll h^\theta \log(1+|t_j|)e^{-\frac{\pi}{2}|t_j|}$ for each *individual* t_j, where θ denotes the best known progress towards the non-achimedian Ramanujan conjecture. This is a superior bound than even the on-average bound given by the Kuznetsof Trace Formula, but it doesn't immediately generalize to half-integral weight Maass forms.

The on-average bounds then give that

$$\langle \mu_j, \theta^{2k+1}\overline{\theta} y^{\frac{k+1}{2}} \rangle \ll (1+|t_j|)^{3k+6+\epsilon} e^{-\frac{\pi}{2}|t_j|}$$

on average. Recalling that there are $O(T^2 \log T)$ Maass forms with $T \leq |t_j| \leq 2T$, summing these terms together gives

$$\sum_{T \leq |t_j| \leq 2T} \rho_j(h) \langle \mu_j, \theta^{2k+1}\overline{\theta} y^{\frac{k+1}{2}} \rangle \ll (1+|t_j|)^{3k+8+\epsilon}.$$

This concludes the proof. □

It is also necessary to note that Selberg's Eigenvalue Conjecture is known for weight k Maass forms on $\Gamma_0(4)$.

Lemma 5.3.9. *Suppose μ_j is a Maass form appearing in the old discrete spectrum described above. Denote the type of μ_j by $\frac{1}{2}+it_j$, so that μ_j is an eigenfunction of the Laplacian with eigenvalue $\lambda = \frac{1}{4}+t_j^2$. Then $\lambda > \frac{1}{4}$. That is, there are no exceptional eigenvalues.*

Proof. This is an argument given by Gergely Harcos at the Alfréd Rényi Institute of Mathematics in an answer at MathOverflow [fMhfm]. We sketch the argument here. A half-integral weight Maass form with eigenvalue $(1-t^2)/4$ has a Shimura lift to an integral weight Maass form on $\Gamma_0(1)$ with eigenvalue $\frac{1}{4}-t^2$. As Selberg's Eigenvalue Conjecture is known in this case (see [BB13]), this completes the proof. □

Now fix $s = \sigma + it$ with $\frac{1}{2} < \sigma < 1$. We write $|t_j| \sim T$ to mean $T \leq |t_j| \leq 2T$ for the rest of this section. Then

$$\sum_{|t_j| \sim T} \langle P_h^k(\cdot, s), \mu_j \rangle \langle \mu_j, V \rangle = \sum_{|t_j| \sim T} \frac{\Gamma(s-\frac{1}{2}+it_j)\Gamma(s-\frac{1}{2}-it_j)}{(4\pi h)^{s-1}\Gamma(s-\frac{k}{2})} \rho_j(h) \langle \mu_j, V \rangle.$$

In this expression it is clear that the rightmost poles in s are at $s = \frac{1}{2} \pm it_j$, which occur on the line $\operatorname{Re} s = \frac{1}{2}$. By Stirling's Approximation, this is asymptotically

$$\sum_{|t_j| \sim T} \frac{(1+|t+t_j|)^{\sigma-1}(1+|t-t_j|)^{\sigma-1}}{(4\pi h)^{\sigma-1+it}(1+|t|)^{\sigma-\frac{1}{2}-\frac{k}{2}}} e^{-\frac{\pi}{2}(|t+t_j|+|t-t_j|-|t|)} \rho_j(h) \langle \mu_j, V \rangle. \tag{3.12}$$

Examination of the exponential contribution shows that there is large exponential decay in t_j when $|t_j| > |t|$, so we only need to investigate the convergence when $|t_j| \leq |t|$. Then (3.12) is bounded by

$$O_h\Big(\sum_{|t_j| \sim T} (1+|t|)^{\frac{k}{2}-\frac{1}{2}} e^{-\frac{\pi}{2}|t|} \rho_j(h) \langle \mu_j, V \rangle \Big),$$

which, by Lemma 5.3.6, is bounded by

$$O_{h,\epsilon}\Big((1+|t|)^{\frac{7}{2}k+\frac{15}{2}+\epsilon} e^{-\frac{\pi}{2}|t|} \Big).$$

Summing dyadically gives the first part of the proposition.

For the second part of the proposition, recall the inner product from Lemma 5.3.4

$$\langle P_h^k(\cdot, s), \mu_{j,\ell} \rangle = \frac{\overline{\rho_{j,\ell}(h)}}{(4\pi h)^{s-1}} \frac{\Gamma(s+\frac{\ell}{2}-1)\Gamma(s-\frac{\ell}{2})}{\Gamma(s-\frac{k}{2})}.$$

As the sum over j and ℓ are each finite for the old discrete spectrum, the analytic properties in s can be read directly from the inner products $\langle P_h^k, \mu_{j,\ell} \rangle$. The leading poles all come from the Gamma function $\Gamma(s-\frac{\ell}{2})$ in the numerator. Note that the first apparent pole is at $s = \frac{k}{2}$ is cancelled by the Gamma function in the denominator, but there are poles at $s = \frac{k}{2} - 1 - m$ for $m \in \mathbb{Z}_{\geq 0}$.

Residual Spectrum

Consider the residual spectrum appearing in line (3.10), which we rewrite for convenience:

$$\sum_{\mathfrak{a}} \langle P_h^k(\cdot, s), R_{\mathfrak{a}}^k \rangle \langle R_{\mathfrak{a}}^k, V \rangle.$$

The residual spectrum only occurs when k is a half-integer. For each cusp, $\langle R_{\mathfrak{a}}^k, V \rangle$ evaluates to some constant and doesn't affect the analysis in s. Referring to Lemma 5.3.4, we see that

$$\langle P_h^k(\cdot, s), R_{\mathfrak{a}} \rangle = \frac{\mathrm{Res}_{w=\frac{3}{4}} \overline{\rho_{\mathfrak{a}}^k(h,w)}}{(4\pi h)^{s-1}} \frac{\Gamma(s-\frac{1}{4})\Gamma(s-\frac{3}{4})}{\Gamma(s-\frac{k}{2})}.$$

As described in (2.1), the residue in $\rho_{\mathfrak{a}}^k(h,w)$ comes from a potential pole in

$$L\Big(2w-\tfrac{1}{2}, \big(\tfrac{h(-1)^{k-\frac{1}{2}}}{\cdot}\big)\Big)$$

at $w = \frac{3}{4}$. This L-function has a pole if and only if the character is trivial, which occurs if and only if h is a square, and $k = \frac{1}{2} + 2m$ for some $m \in \mathbb{Z}_{\geq 0}$.

Therefore, if h is not a square or if k is not of the form $\frac{1}{2}+2m$, then the residual spectrum vanishes. If h is a square and $k=\frac{1}{2}+2m$, then the residue is nonzero, and the analytic properties of the residual spectrum can be read directly from

$$\frac{\Gamma(s-\frac{1}{4})\Gamma(s-\frac{3}{4})}{(4\pi h)^{s-1}\Gamma(s-\frac{k}{2})}. \tag{3.13}$$

We codify this in a proposition.

Proposition 5.3.10. *The residual spectrum in line (3.10) vanishes unless h is a square and $k=\frac{1}{2}+2m$ for some $m\in\mathbb{Z}_{\geq 0}$, in which case the residual spectrum has meromorphic continuation to the plane and is analytic for $\operatorname{Re} s>0$, except possibly for poles at $s=\frac{3}{4}$ and $s=\frac{1}{4}$.*

Continuous Spectrum

Consider the continuous spectrum appearing in line (3.11), which we rewrite for convenience:

$$\frac{1}{4\pi i}\sum_{\mathfrak{a}}\int_{(\frac{1}{2})}\langle P_h^k(\cdot,s),E_{\mathfrak{a}}^k(\cdot,u)\rangle\langle E_{\mathfrak{a}}^k(\cdot,u),V\rangle du.$$

Referring to Lemma 5.3.4, we see that

$$\langle P_h(\cdot,s),E_{\mathfrak{a}}^k(\cdot,\tfrac{1}{2}+it)\rangle=\frac{\overline{\rho_{\mathfrak{a}}(h,\frac{1}{2}+it)}}{(4\pi h)^{s-1}}\frac{\Gamma(s-\frac{1}{2}+it)\Gamma(s-\frac{1}{2}-it)}{\Gamma(s-\frac{k}{2})}.$$

The Gamma functions can be approximated through Stirling's Approximation. Each $\rho_{\mathfrak{a}}(h,\frac{1}{2}+it)$ can understood through the Phragmén-Lindelöf Principle to satisfy the bound $\rho_{\mathfrak{a}}(h,\frac{1}{2}+it)\ll(1+|t|)^{\frac{1}{4}-\frac{k}{2}+\epsilon}$.

The first apparent poles in s can be read from the Gamma functions, and it is clear that there are no poles in s for $\operatorname{Re} s>\frac{1}{2}$.

The other inner products, $\langle E_{\mathfrak{a}}^k(\cdot,u),V\rangle$, can be understood through Zagier normalization [Zag81]. In particular, since $V=cy^{\frac{1-k}{2}}+O(y^{-N})$ for some constant c and any $N\geq 0$ as $y\to\infty$ (and more generally, at each cusp), Zagier normalization allows us to identify

$$\langle E_\infty^k(\cdot,u),V\rangle=\int_0^\infty\widetilde{V}_0(y)y^{u-1}\frac{dy}{y}=\frac{\Gamma(s+\frac{k+1}{2}-1)}{(4\pi)^{u+\frac{k+1}{2}-1}}\sum_{n\geq 1}\frac{r_{2k+1}(n)r_1(n)}{n^{u+\frac{k+1}{2}-1}} \tag{3.14}$$

for $\frac{1-k}{2}<\operatorname{Re} u<\frac{k+1}{2}$ and give meromorphic continuation to the plane. Here, $\widetilde{V}_0$ is the 0th Fourier coefficient of $\widetilde{V}=\theta^{2k+1}\overline{\theta}$, which was first defined in §5.1. Notice that the region of identification includes $\operatorname{Re} u=\frac{1}{2}$, as is necessary for this application. Similar expressions exist at the cusps 0 and $\frac{1}{2}$.

Remark 5.3.11. Zagier normalization also gives that the Dirichlet series at the right in (3.14) has a potential pole at $u = \frac{k+1}{2}$, which agrees with on-average estimates. The function $r_1(n)$ is essentially a square indicator function, so the Dirichlet series can be rewritten as

$$2\sum_{n\geq 1}\frac{r_{2k+1}(n^2)}{n^{2(u+\frac{k+1}{2}-1)}},$$

in which it is straightforward to use the Gaussian heuristic to confirm the pole from Zagier normalization.

Using the functional equation of the Eisenstein series to give the functional equation of the Dirichlet series in (3.14) and applying the Phragmén-Lindelöf Convexity Principle guarantees that

$$\langle E_{\mathfrak{a}}^k(\cdot, \tfrac{1}{2}+it), V\rangle \ll (1+|t|)^{2k-1+\epsilon}.$$

Denote $\operatorname{Re} s = \sigma$, and suppose $\frac{1}{2} < \sigma < 1$. Applying Stirling's Approximation and the bounds above, we can now estimate

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(\frac{1}{2})}\langle P_h^k(\cdot,s), E_{\mathfrak{a}}^k(\cdot,u)\rangle\langle E_{\mathfrak{a}}^k(\cdot,u), V\rangle du\\ &\quad\ll \int_{-\infty}^{\infty}\frac{(1+|t|)^{(\frac{1}{4}-\frac{k}{2}+\epsilon)+(2k-1+\epsilon)}}{(4\pi h)^{\sigma-1}}\frac{(1+|s+t|)^{\sigma-1}(1+|s-t|)^{\sigma-1}}{(1+|s|)^{\sigma-\frac{1}{2}-\frac{k}{2}}}e^{-\frac{\pi}{2}(|s-t|+|s+t|-|s|)}dt.\end{aligned}$$

When $|t| > |s|$, there is significant exponential decay in t, effectively cutting off the integral to the interval $|t| \leq |s|$. Within this interval, the integral can be bounded by

$$\int_{-|s|}^{|s|}(1+|t|)^{\frac{3}{2}k-\frac{3}{4}+\epsilon}(1+|s|)^{\frac{k}{2}-\frac{1}{2}}e^{-\frac{\pi}{2}|s|}dt \ll (1+|s|)^{2k-\frac{1}{4}+\epsilon}e^{-\frac{\pi}{2}|s|}.$$

Proposition 5.3.12. *The continuous spectrum in line (3.11) has meromorphic continuation to the plane and is analytic for* $\operatorname{Re} s > \frac{1}{2}$ *and has apparent poles at* $\operatorname{Re} s = \frac{1}{2}$. *For* $\frac{1}{2} < \operatorname{Re} s < 1$, *the continuous spectrum satisfies the bound*

$$\frac{1}{4\pi i}\sum_{\mathfrak{a}}\int_{(\frac{1}{2})}\langle P_h^k(\cdot,s), E_{\mathfrak{a}}^k(\cdot,u)\rangle\langle E_{\mathfrak{a}}^k(\cdot,u), V\rangle du \ll (1+|s|)^{2k-\frac{1}{4}+\epsilon}e^{-\frac{\pi}{2}|s|}.$$

5.3.4 Analytic Behavior of $D_h^k(s)$

We are now ready to describe the analytic behavior of $D_h^k(s)$ for each $k \geq \frac{1}{2}$, for $\operatorname{Re} s > \frac{1}{2}$. Recall that $D_h^k(s)$ is defined in (3.1) and (3.4) as

$$D_h^k(s) = \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})}\langle P_h^{\frac{1}{2}}(\cdot,s), V\rangle.$$

In Propositions 5.3.1 (for $k \geq 1$) and 5.3.3 (for $k = \frac{1}{2}$), it was shown that

$$D_h^k(s) = \sum_{m \in \mathbb{Z}} \frac{r_{2k+1}(m^2+h)}{(2m^2+h)^{s+\frac{k-1}{2}}} - \mathfrak{E}_h^k(s) \tag{3.15}$$

and the analytic properties of $\mathfrak{E}_h^k(s)$ are described. On the other hand, through the Spectral Expansion of P_h^k, we also have an expression for $\langle P_h^k(\cdot, s), V\rangle$, given in (3.9)–(3.11). Multiplying by $(2\pi)^{s+\frac{k-1}{2}}\Gamma(s+\frac{k-1}{2})^{-1}$ and rearranging (3.15), we have

$$\begin{aligned}
\sum_{m \in \mathbb{Z}} \frac{r_{2k+1}(m^2+h)}{(2m^2+h)^{s+\frac{k-1}{2}}} &= \mathfrak{E}_h^k(s) \\
&+ \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})}\left(\sum_j \langle P_h^k, \mu_j\rangle\langle \mu_j, V\rangle + \sum_{\ell,j}\langle P_h^k, \mu_{j,\ell}\rangle\langle \mu_{j,\ell}, V\rangle\right) \\
&+ \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})}\sum_{\mathfrak{a}}\langle P_h^k, R_{\mathfrak{a}}\rangle\langle R_{\mathfrak{a}}, V\rangle \\
&+ \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})}\frac{1}{4\pi i}\sum_{\mathfrak{a}}\int_{(\frac{1}{2})}\langle P_h^k, E_{\mathfrak{a}}^k\rangle\langle E_{\mathfrak{a}}^k, V\rangle,
\end{aligned} \tag{3.16}$$

where the lines are separated into the Eisenstein correction factors in V, the discrete spectrum, the residual spectrum, and the continuous spectrum, respectively. The analytic properties of the discrete, residual, and continuous spectra are described in Propositions 5.3.5, 5.3.10, and 5.3.12, respectively. Assembling these propositions together, we have proved the following theorem.

Theorem 5.3.13. *Let $h \geq 1$ be an integer and $k \geq \frac{1}{2}$ be either an integer or a half-integer. The Dirichlet series*

$$\sum_{m \in \mathbb{Z}} \frac{r_{2k+1}(m^2+h)}{(2m^2+h)^{s+\frac{k-1}{2}}}$$

has meromorphic continuation to the plane, and is analytic for $\operatorname{Re} s > \frac{1}{2}$ except for

- *simple poles at $s = \frac{k}{2} - 1 - m$ for $m \in \mathbb{Z}_{\geq 0}$, coming from the discrete spectrum,*
- *simple poles at $s = \frac{k+1}{2} - m$ for $m \in \mathbb{Z}_{\geq 0}$, coming from the Eisenstein correction factors $\mathfrak{E}_h^k(s)$, and*
- *a double pole at $s = \frac{3}{4}$ when $k = \frac{1}{2}$ and h is a square, also coming from $\mathfrak{E}_h^k(s)$.*

5.4 Integral Analysis

We are now ready to perform the main integral analysis on $\langle P_h^k(\cdot, s), V\rangle$. In this section, we handle the $k > \frac{1}{2}$ case. We will examine

$$\frac{1}{2\pi i}\int_{(\sigma)} \sum_{n\in\mathbb{Z}} \frac{r_{2k+1}(n^2+h)}{(2n^2+h)^{s+\frac{k-1}{2}}} X^{s+\frac{k-1}{2}} V_Y(s)ds,$$

which is closely related to studying

$$\frac{1}{2\pi i}\int_{(\sigma)} \langle P_h^k(\cdot, s), V\rangle \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})} X^{s+\frac{k-1}{2}} V_Y(s)ds.$$

We will use three integral kernels $V_Y(s)$ described in Chapter 2. We denote the Mellin transform of $V_Y(s)$ by $v_Y(x)$ when appropriate.

From the decomposition in (3.16), it will suffice to consider the integral transforms applied to $\mathfrak{E}_h^k$, the discrete spectrum, the residual spectrum, and the continuous spectrum separately. In each integral, we will shift the line of integration to $\frac{1}{2}+\epsilon$ for a small $\epsilon > 0$ and analyze the poles and residues.

5.4.1 Integral Analysis of $\mathfrak{E}_h^k$

We first study

$$\frac{1}{2\pi i}\int_{(\sigma)} \mathfrak{E}_h^k(s) X^{s+\frac{k-1}{2}} V_Y(s)ds.$$

From Theorem 5.3.13 and Propositions 5.3.1 and 5.3.3, we recognize that $\mathfrak{E}_h^k$ has poles at $s = \frac{k+1}{2} - m$ for $m \in \mathbb{Z}_{\geq 0}$. All of these poles are simple, except when $k = \frac{1}{2}$ and h is a square, in which case the leading pole at $s = \frac{k+1}{2} = \frac{3}{4}$ is a double pole. As $\mathfrak{E}_h^k$ is of moderate growth for $\operatorname{Re} s > \frac{1}{2}$ and each kernel $V_Y(s)$ is of rapid decay, we may shift the line of integration to $\frac{1}{2}+\epsilon$, and by Cauchy's Theorem we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(\sigma)} \mathfrak{E}_h^k(s) X^{s+\frac{k-1}{2}} V_Y(s)ds \\
&\quad= \frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)} \mathfrak{E}_h^k(s) X^{s+\frac{k-1}{2}} V_Y(s)ds + \sum_{0\leq m<\frac{k}{2}} R_{k-m,h}^k X^{k-m} V_Y(\tfrac{k+1}{2}-m) \\
&\qquad+ \delta_{[k=\frac{1}{2}]}\delta_{[h=a^2]}\left(R_h' X^{\frac{1}{2}} \log X V_Y(\tfrac{3}{4}) + R_h' X^{\frac{1}{2}} V_Y'(\tfrac{3}{4})\right)
\end{aligned} \tag{4.1}$$

where

$$R_{k-m,h}^k = \operatorname*{Res}_{s=\frac{k+1}{2}-m} \mathfrak{E}_h^k(s)$$

and where R'_h is the coefficient of $(s-\frac{3}{4})^{-2}$ in the Laurent expansion of $\mathfrak{E}_h^{\frac{1}{2}}(s)$. The Kronecker δ symbol is used here to mean

$$\delta_{[k=\frac{1}{2}]}\delta_{[h=a^2]} = \begin{cases} 1 & \text{if } k=\frac{1}{2} \text{ and } h \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

To estimate the shifted integral, recall from Propositions 5.3.1 and 5.3.3 that for $\operatorname{Re} s > \frac{1}{2}$, we know the bound $\mathfrak{E}_h^k(s) \ll (1+|s|)^{\frac{1}{2}}$. Therefore as long as $V_Y(s) \ll (1+|s|)^{-\frac{3}{2}-\epsilon}$, the integral converges absolutely. With respect to the three integral transforms, this means that

$$\frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)} \mathfrak{E}_h^k(s) X^{s+\frac{k-1}{2}} \Gamma(s+\tfrac{k-1}{2}) ds \ll X^{\frac{k}{2}+\epsilon} \tag{4.2}$$

$$\frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)} \mathfrak{E}_h^k(s) X^{s+\frac{k-1}{2}} \frac{e^{\pi s^2/y^2}}{y} ds \ll X^{\frac{k}{2}+\epsilon} Y^{\frac{1}{2}} \tag{4.3}$$

$$\frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)} \mathfrak{E}_h^k(s) X^{s+\frac{k-1}{2}} \Phi_Y(s) ds \ll X^{\frac{k}{2}+\epsilon} Y^{\frac{1}{2}+\epsilon}. \tag{4.4}$$

5.4.2 Integral analysis of the discrete spectrum

We now study the integral of the discrete spectrum. To condense notation, we introduce the notation

$$\operatorname{discrete}_h^k(s) := \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})}\left(\sum_j \langle P_h^k, \mu_j\rangle\langle \mu_j, V\rangle + \sum_{\ell,j}\langle P_h^k, \mu_{j,\ell}\rangle\langle \mu_{j,\ell}, V\rangle\right).$$

Then the integral of the discrete spectrum can be written as

$$\frac{1}{2\pi i}\int_{(\sigma)} \operatorname{discrete}_h^k(s) X^{s+\frac{k-1}{2}} V_Y(s) ds. \tag{4.5}$$

From Theorem 5.3.13 we recognize that the integrand has simple poles at $s=\frac{k}{2}-m$ for $m \in \mathbb{Z}_{\geq 0}$, coming from the finitely many terms of the "old" discrete spectrum. Therefore, shifting the line of integration to $\frac{1}{2}+\epsilon$ and applying Cauchy's Theorem shows that (4.5) is equal to

$$\sum_{0\leq m\leq \frac{k-1}{2}} R^k_{k-\frac{1}{2}-m,h} X^{k-\frac{1}{2}-m} V(\tfrac{k+1}{2}-m-\tfrac{1}{2}) + \frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)} \operatorname{discrete}_h^k(s) X^{s+\frac{k-1}{2}} V_Y(s) ds,$$

where $R^k_{k-\frac{1}{2}-m}$ are the collected residues of the old discrete spectrum at $s=\frac{k}{2}-m$.

To estimate the shifted integral, note that from applying Stirling's Approximation to $\Gamma(s+\frac{k-1}{2})^{-1}$ and using the approximations from Proposition 5.3.5 that $\text{discrete}_h^k(s) \ll (1+|s|)^{3k+\frac{17}{2}+\epsilon}$. Therefore as long as $V_Y(s) \ll (1+|s|)^{-3k-\frac{19}{2}-2\epsilon}$, the integral converges absolutely. The three integral transforms then give

$$\frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)} \text{discrete}_h^k(s) X^{s+\frac{k-1}{2}} \Gamma(s+\frac{k-1}{2}) ds \ll X^{\frac{k}{2}+\epsilon} \tag{4.6}$$

$$\frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)} \text{discrete}_h^k(s) X^{s+\frac{k-1}{2}} \frac{e^{\pi} s^2/Y^2}{Y} ds \ll X^{\frac{k}{2}+\epsilon} Y^{3k+\frac{17}{2}+\epsilon} \tag{4.7}$$

$$\frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)} \text{discrete}_h^k(s) X^{s+\frac{k-1}{2}} \Phi_Y(s) ds \ll X^{\frac{k}{2}+\epsilon} Y^{3k+\frac{17}{2}+2\epsilon}. \tag{4.8}$$

5.4.3 Integral analysis of the residual spectrum

We now study the integral of the residual spectrum,

$$\frac{1}{2\pi i}\int_{(\sigma)} \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})} \sum_{\mathfrak{a}} \langle P_h^k, R_{\mathfrak{a}}^k\rangle \langle R_{\mathfrak{a}}^k, V\rangle X^{s+\frac{k-1}{2}} V_Y(s) ds. \tag{4.9}$$

From Theorem 5.3.13, Proposition 5.3.10, and the analytic properties of the residual spectrum as described in (3.13), the residual spectrum is analytic for $\operatorname{Re} s > \frac{1}{2}+\epsilon$ and is bounded by $O((1+|s|)^{\frac{1}{2}})$. By Cauchy's Theorem the integral (4.9) is equal to the shifted integral

$$\frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)} \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})} \sum_{\mathfrak{a}} \langle P_h^k, R_{\mathfrak{a}}\rangle \langle R_{\mathfrak{a}}^k, V\rangle X^{s+\frac{k-1}{2}} V_Y(s) ds.$$

As long as $V_Y(s) \ll (1+|s|)^{-\frac{3}{2}+\epsilon}$, this converges absolutely. The three integral transforms then satisfy the bounds

$$\frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)} \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})} \sum_{\mathfrak{a}} \langle P_h^k, R_{\mathfrak{a}}^k\rangle \langle R_{\mathfrak{a}}^k, V\rangle X^{s+\frac{k-1}{2}} \Gamma(s+\tfrac{k-1}{2}) ds \ll X^{\frac{k}{2}+\epsilon} \tag{4.10}$$

$$\frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)} \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})} \sum_{\mathfrak{a}} \langle P_h^k, R_{\mathfrak{a}}^k\rangle \langle R_{\mathfrak{a}}^k, V\rangle X^{s+\frac{k-1}{2}} \frac{e^{\pi s^2/Y^2}}{Y} ds \ll X^{\frac{k}{2}+\epsilon} Y^{\frac{1}{2}+\epsilon} \tag{4.11}$$

$$\frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)} \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})} \sum_{\mathfrak{a}} \langle P_h^k, R_{\mathfrak{a}}^k\rangle \langle R_{\mathfrak{a}}^k, V\rangle X^{s+\frac{k-1}{2}} \Phi_Y(s) ds \ll X^{\frac{k}{2}+\epsilon} Y^{\frac{1}{2}+2\epsilon}. \tag{4.12}$$

5.4.4 Integral analysis of the continuous spectrum

We now consider the last integral, the integral of the continuous spectrum:

$$\frac{1}{2\pi i}\int_{(\sigma)}\frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})}\frac{1}{4\pi i}\sum_{\mathfrak{a}}\int_{(\frac{1}{2})}\langle P_h^k, E_{\mathfrak{a}}^k\rangle\langle E_{\mathfrak{a}}^k, V\rangle X^{s+\frac{k-1}{2}}V_Y(s)du\,ds. \tag{4.13}$$

Recall that we use u to denote the variable within the Eisenstein series $E_{\mathfrak{a}}^k(z,u)$, though we omit this from the notation. By Theorem 5.3.13, we know that the continuous spectrum is analytic for $\operatorname{Re} s > \frac{1}{2}$. By Cauchy's Theorem, the integral (4.13) is equal to the shifted integral

$$\frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)}\frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})}\frac{1}{4\pi i}\sum_{\mathfrak{a}}\int_{(\frac{1}{2})}\langle P_h^k, E_{\mathfrak{a}}^k\rangle\langle E_{\mathfrak{a}}^k, V\rangle X^{s+\frac{k-1}{2}}V_Y(s)du\,ds.$$

From Stirling's Approximation applied to $\Gamma(s+\frac{k-1}{2})^{-1}$ and the bounds from Proposition 5.3.12, as long as $V_Y(s) \ll (1+|s|)^{-\frac{3}{2}k-\frac{7}{4}-2\epsilon}$, the shifted integral converges absolutely. The three integral transforms thus satisfy the bounds

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)}\frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})}\frac{1}{4\pi i}\\&\quad\times\sum_{\mathfrak{a}}\int_{(\frac{1}{2})}\langle P_h^k, E_{\mathfrak{a}}^k\rangle\langle E_{\mathfrak{a}}^k, V\rangle X^{s+\frac{k-1}{2}}\Gamma(s+\frac{k-1}{2})du\,ds \ll X^{\frac{k}{2}+\epsilon}\end{aligned} \tag{4.14}$$

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)}\frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})}\frac{1}{4\pi i}\\&\quad\times\sum_{\mathfrak{a}}\int_{(\frac{1}{2})}\langle P_h^k, E_{\mathfrak{a}}^k\rangle\langle E_{\mathfrak{a}}^k, V\rangle X^{s+\frac{k-1}{2}}\frac{e^{\pi s^2/Y^2}}{Y}du\,ds \ll X^{\frac{k}{2}+\epsilon}Y^{\frac{3}{2}k+\frac{3}{4}+\epsilon}\end{aligned} \tag{4.15}$$

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(\frac{1}{2}+\epsilon)}\frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma(s+\frac{k-1}{2})}\frac{1}{4\pi i}\\&\quad\times\sum_{\mathfrak{a}}\int_{(\frac{1}{2})}\langle P_h^k, E_{\mathfrak{a}}^k\rangle\langle E_{\mathfrak{a}}^k, V\rangle X^{s+\frac{k-1}{2}}\Phi_Y(s)du\,ds \ll X^{\frac{k}{2}+\epsilon}Y^{\frac{3}{2}k+\frac{3}{4}+2\epsilon}.\end{aligned} \tag{4.16}$$

5.5 Proof of Main Theorems

Using the integral analysis from the previous section, we now prove the main theorems of this chapter.

5.5.1 Smoothed Main Theorem

Consider the integral transform

$$\frac{1}{2\pi i}\int_{(\sigma)}\sum_{m\in\mathbb{Z}}\frac{r_{2k+1}(m^s+h)}{(2m^2+h)^{s+\frac{k-1}{2}}}X^{s+\frac{k-1}{2}}\Gamma(s+\tfrac{k-1}{2})ds.$$

On the one hand, by the standard properties of the transform (in §2.3), this is exactly the sum

$$\sum_{m\in\mathbb{Z}} r_{2k+1}(m^2+h)e^{-(2m^2+h)/X}.$$

On the other hand, by (3.16) and the analysis in the previous section (and in particular the bounds from lines (4.2), (4.6), (4.10), and (4.14), as well as the residual expression in (4.1)), this transform is equal to

$$\sum_{0\le m<\frac{k}{2}} R^k_{k-m,h}\Gamma(k-m)X^{k-m}+\delta_{[k=\frac{1}{2}]}\delta_{[h=a^2]}\left(R'_hX^{\frac{1}{2}}\log X\Gamma(\tfrac{1}{2})+R'_h\Gamma'(\tfrac{1}{2})X^{\frac{1}{2}}\right)$$
$$+\sum_{0\le m<\frac{k-1}{2}} R^k_{k-\frac{1}{2}-m,h}\Gamma(k-\tfrac{1}{2}-m)X^{k-\frac{1}{2}-m}+O(X^{\frac{k}{2}+\epsilon}).$$

This proves the following theorem.

Theorem 5.5.1. *Let $k\ge\frac{1}{2}$ be a full or half-integer. Then*

$$\sum_{m\in\mathbb{Z}} r_{2k+1}(m^2+h)e^{-(2m^2+h)/X}=$$
$$=\delta_{[k=\frac{1}{2}]}\delta_{[h=a^2]}\left(R'_hX^{\frac{1}{2}}\log X\Gamma(\tfrac{1}{2})+R'_h\Gamma'(\tfrac{1}{2})X^{\frac{1}{2}}\right)+\sum_{0\le m<\frac{k}{2}} R^k_{k-m,h}\Gamma(k-m)X^{k-m}$$
$$+\sum_{0\le m<\frac{k-1}{2}} R^k_{k-\frac{1}{2}-m,h}\Gamma(k-\tfrac{1}{2}-m)X^{k-\frac{1}{2}-m}+O(X^{\frac{k}{2}+\epsilon}).$$

Here, $\delta_{[condition]}$ is a Kronecker δ and evaluates to 1 if the condition is true and 0 otherwise, and the constants $R^k_{\ell,h}$ and R'_h are residues as defined in §5.4.

5.5.2 Main Theorem in Short-Intervals

Consider the integral transform

$$\frac{1}{2\pi i}\int_{(\sigma)}\sum_{m\in\mathbb{Z}}\frac{r_{2k+1}(m^s+h)}{(2m^2+h)^{s+\frac{k-1}{2}}}X^{s+\frac{k-1}{2}}\frac{e^{\pi s^2/Y^2}}{Y}ds.$$

On the one hand, by the properties of this integral transform as described in §2.3, this is exactly the sum

$$\sum_{m\in\mathbb{Z}} r_{2k+1}(m^2+h)\exp\Big(-\frac{Y^2\log^2(X/(2m^2+h))}{4\pi}\Big).$$

When $2m^2+h\in[X-X/Y,X+X/Y]$, the exponential damping term is almost constant. But for m with $|2m^2+h-X|>X/Y^{1-\epsilon}$, the exponential term contributes significant exponential decay. Further, by the positivity of r_{2k+1}, we have that

$$\sum_{|2m^2+h-X|<\frac{X}{Y}} r_{2k+1}(m^2+h)\ll\frac{1}{2\pi i}\int_{(\sigma)}\sum_{m\in\mathbb{Z}}\frac{r_{2k+1}(m^s+h)}{(2m^2+h)^{s+\frac{k-1}{2}}}X^{s+\frac{k-1}{2}}\frac{e^{\pi s^2/Y^2}}{Y}ds.$$

Remark 5.5.2. Roughly speaking, this should be interpreted to mean that this integral transform concentrates the mass of the integral on those m such that $2m^2+h$ is within a short interval around X.

And on the other hand, by (3.16) and the analysis in the previous section (in particular the bounds from lines (4.3), (4.7), (4.11), and (4.15), as well as the residual expression in (4.1)), this transform is equal to

$$\delta_{[k=\frac{1}{2}]}\delta_{[h=a^2]}\Big(R_h'X^{\frac{1}{2}}\log X+R_h'X^{\frac{1}{2}}\frac{\pi}{Y}\Big)\frac{\exp(\frac{\pi}{4Y})}{Y}+R_k^kX^k\frac{\exp(\frac{\pi k^2}{Y})}{Y}$$
$$+O(\frac{X^{k-\frac{1}{2}}}{Y})+O(X^{\frac{k}{2}+\epsilon}Y^{3k+\frac{17}{2}+\epsilon}).$$

In this expression, we only kept the leading poles. As we are only seeking to create an upper bound, we simplify the above expression into the bound

$$O(\frac{X^{k+\epsilon}}{Y})+O(X^{\frac{k}{2}+\epsilon}Y^{3k+\frac{17}{2}+\epsilon}). \tag{5.1}$$

We choose Y to balance the expressions in (5.1). The two terms are balaned when $Y=X^{1/(6+\frac{19}{k}+\frac{\epsilon}{k})}$. This gives the overall bound

$$O(X^{k+\epsilon-\lambda(k)}),\qquad\text{where }\lambda(k)=\frac{1}{6+\frac{19}{k}}. \tag{5.2}$$

We have now shown that

$$\sum_{|2m^2+h-X|<X^{1+\epsilon-\lambda(k)}} r_{2k+1}(m^2+h)\ll X^{k+\epsilon-\lambda(k)},$$

where $\lambda(k)$ is defined by (5.2). This is the content of the second main theorem in this chapter.

Theorem 5.5.3. *Let $k \geq \frac{1}{2}$ be a full or half-integer. Then*

$$\sum_{|2m^2+h-X|<X^{1+\epsilon-\lambda(k)}} r_{2k+1}(m^2+h) \ll X^{k+\epsilon-\lambda(k)},$$

where $\lambda(k)$ is defined as

$$\lambda(k) = \frac{1}{6+\frac{19}{k}}.$$

5.5.3 Sharp Main Theorem

Finally, consider the integral transform

$$\frac{1}{2\pi i}\int_{(\sigma)} \sum_{m\in\mathbb{Z}} \frac{r_{2k+1}(m^s+h)}{(2m^2+h)^{s+\frac{k-1}{2}}} X^{s+\frac{k-1}{2}} \Phi_Y(s) ds.$$

On the one hand, by the description of this integral transform in §2.3, this is exactly the sum

$$\sum_{|2m^2+h|\leq X} r_{2k+1}(m^2+h) + \sum_{X\leq|2m^2+h|\leq X+\frac{X}{Y}} r_{2k+1}(m^2+h)\phi_Y(\tfrac{2m^2+h}{X}). \tag{5.3}$$

As r_{2k+1} is always positive, we can bound the second term above by

$$\sum_{X\leq|2m^2+h|\leq X+\frac{X}{Y}} r_{2k+1}(m^2+h)\phi_Y(\tfrac{2m^2+h}{X}) \ll \sum_{|2m^2+h-X|\leq\frac{X}{Y}} r_{2k+1}(m^2+h). \tag{5.4}$$

Notice that this is a short-interval type estimate, exactly as considered in Theorem 5.5.3.

On the other hand, by (3.16) and the analysis in the previous section (and in particular the bounds from lines (4.4), (4.8), (4.12), and (4.16), as well as the residual expression in (4.1)), this integral transform is equal to

$$\begin{aligned}
&\sum_{0\leq m<\frac{k}{2}} R^k_{k-m,h} X^{k-m}\big(\tfrac{1}{k-m}+O(\tfrac{1}{Y})\big) \\
&\quad + \delta_{[k=\frac{1}{2}]}\delta_{[h=a^2]}\Big(R'_h X^{\frac{1}{2}}\log X\big(2+O(\tfrac{1}{Y})\big) + R'_h X^{\frac{1}{2}}\big(-4+O(\tfrac{1}{Y})\big)\Big) \\
&\quad + \sum_{0\leq m<\frac{k-1}{2}} R^k_{k-\frac{1}{2}-m,h} X^{k-\frac{1}{2}-m}\big(\tfrac{1}{k-\frac{1}{2}-m}+O(\tfrac{1}{Y})\big) \\
&\quad + O(X^{\frac{k}{2}+\epsilon}Y^{\frac{1}{2}+\epsilon}) + O(X^{\frac{k}{2}+\epsilon}Y^{3k+\frac{17}{2}+2\epsilon}) + O(X^{\frac{k}{2}+\epsilon}Y^{\frac{3}{2}k+\frac{3}{4}+2\epsilon}).
\end{aligned}$$

We note that we have used that $\Phi_Y(s) = \frac{1}{s} + O(\frac{1}{Y})$ and $\Phi'_Y(s) = -\frac{1}{s^2} + O(\frac{1}{Y})$ from §2.3 to simplify the residual terms involving the weight function Φ_Y. These contribute to the error terms in this expression.

Keeping only the leading terms of growth, we rewrite this as

$$\delta_{[k=\frac{1}{2}]}\delta_{[h=a^2]}\left(2R'_h X^{\frac{1}{2}}\log X - 4R'_h X^{\frac{1}{2}}\right) + \tfrac{1}{k}R^k_{k,h}X^k + O\big(\frac{X^{\frac{1}{2}}\log X}{Y}\big) + O\big(\frac{X^k}{Y}\big) + O(X^{\frac{k-1}{2}}) + O(X^{\frac{k}{2}+\epsilon}Y^{3k+\frac{17}{2}+2\epsilon}).$$

The collected error terms can be written as

$$O(X^{\frac{k-1}{2}}) + O\big(\frac{X^{k+\epsilon}}{Y}\big) + O(X^{\frac{k}{2}+\epsilon}Y^{3k+\frac{17}{2}+2\epsilon})$$

Notice that the terms including Y are the exact same expression as in (5.1), and so the choice of Y that optimizes the error bound is the same! That is, we choose $Y = X^{\lambda(k)}$ where $\lambda(k)$ is defined as in Theorem 5.5.3.

By combining this optimal error term and choice of $Y = X^{\lambda(k)}$ with the expression (5.3) and the bound (5.4), we have shown that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(\sigma)}\sum_{m\in\mathbb{Z}}\frac{r_{2k+1}(m^s+h)}{(2m^2+h)^{s+\frac{k-1}{2}}}X^{s+\frac{k-1}{2}}\Phi_Y(s)ds \\
&\quad= \sum_{|2m^2+h|\le X} r_{2k+1}(m^2+h) + O\Bigg(\sum_{|2m^2+h-X|\le X^{1+\epsilon-\lambda(k)}} r_{2k+1}(m^2+h)\Bigg) \\
&\quad= \delta_{[k=\frac{1}{2}]}\delta_{[h=a^2]}\left(2R'_h X^{\frac{1}{2}}\log X - 4R'_h X^{\frac{1}{2}}\right) + \tfrac{1}{k}R^k_{k,h}X^k + O(X^{k+\epsilon-\lambda(k)}).
\end{aligned}$$

Rearranging, this shows that

$$\begin{aligned}
&\sum_{|2m^2+h|\le X} r_{2k+1}(m^2+h) \\
&\quad= \delta_{[k=\frac{1}{2}]}\delta_{[h=a^2]}\left(2R'_h X^{\frac{1}{2}}\log X - 4R'_h X^{\frac{1}{2}}\right) + \tfrac{1}{k}R^k_{k,h}X^k + O(X^{k+\epsilon-\lambda(k)}) \\
&\quad+ O\Bigg(\sum_{|2m^2+h-X|\le X^{1+\epsilon-\lambda(k)}} r_{2k+1}(m^2+h)\Bigg).
\end{aligned}$$

By Theorem 5.5.3, the last big Oh term is bounded by $O(X^{k+\epsilon-\lambda(k)})$. This concludes the proof of the following theorem.

Theorem 5.5.4. *Let $k \ge \frac{1}{2}$ be a full or half-integer. Then*

$$\sum_{|2m^2+h-X|\le X} r_{2k+1}(m^2+h)$$

$$= \delta_{[k=\frac{1}{2}]}\delta_{[h=a^2]}\left(2R'_h X^{\frac{1}{2}} \log X - 4R'_h X^{\frac{1}{2}}\right) + \tfrac{1}{k} R^k_{k,h} X^k + O(X^{k+\epsilon-\lambda(k)}),$$

where $\lambda(k)$ *is defined as*

$$\lambda(k) = \frac{1}{6 + \frac{19}{k}}.$$

In particular, if $k = \frac{1}{2}$ *then*

$$\sum_{|2m^2+h-X|\leq X} r_{2k+1}(m^2+h)$$
$$= \delta_{[h=a^2]}\left(2R'_h X^{\frac{1}{2}} \log X - 4R'_h X^{\frac{1}{2}}\right) + 2R^{1/2}_{1/2,h} X^{\frac{1}{2}} + O(X^{\frac{1}{2}-\frac{1}{44}+\epsilon}),$$

and for $k \geq 1$

$$\sum_{|2m^2+h-X|\leq X} r_{2k+1}(m^2+h) = \tfrac{1}{k} R^k_{k,h} X^k + O(X^{k+\epsilon-\lambda(k)}).$$

Remark 5.5.5. It is interesting that the shape of the main term is different in the dimension 3 case (when $k = \frac{1}{2}$) compared to all higher dimensions. There is a rough heuristic argument that explains this. Counting solutions to $X^2 + Y^2 = Z^2 + h$ is the same as counting solutions to $(Z - X)(Z + X) = Y^2 - h$, which can be thought of as counting the number of factorizations of $Y^2 - h$ as Y varies. The number of factorizations of $Y^2 - h$ depends heavily on whether or not h is a square. If h is not a square, then there are expected to be relatively few factorizations. If h is a square, then there are expected to be logarithmically many factorizations in Y. Thus if Y varies up to size $\sqrt{X}$ and there are $\log Y$ many factorizations on average, we should expect $X^{\frac{1}{2}} \log X$ terms.

In higher dimensions, this factorization heuristic doesn't apply. And in higher dimensions, the regularity of representation of integers as sums of many squares should smooth away irregularities present in low dimension.

Chapter Six

An Application of Counting Lattice Points on Hyperboids

In this Chapter, we highlight one particular application of counting lattice points on one-sheeted hyperboloids. In particular, we describe how to understand sums of the form

$$\sum_{n \leq X} d(n^2+1),$$

where $d(n)$ denotes the number of divisors of n.

Connection to the Divisor Sum $\sum d(n^2+1)$

The special three dimensional hyperboloid

$$X^2 + Y^2 = Z^2 + 1$$

is closely related to the divisor sum

$$\sum_{n \leq R} d(n^2+1),$$

which has been heavily studied by Hooley [Hoo63]. This relationship is visible through the following theorem.

Theorem 6.0.1. *Let $d_o(n)$ denote the number of positive odd divisors of n. Then*

$$\#\{\textit{Integer points on } X^2 + Y^2 = Z^2 + 1 \textit{ with } |Z| \leq R\} = \sum_{n \leq R} 4 d_o(n^2+1).$$

To prove this theorem, we first prove this lemma.

Lemma 6.0.2. *Given an integer Z, we first show that Z^2+1 is not divisible by any prime congruent to* 3 mod 4.

Proof. Indeed, factorize $Z + i$ as a product of Gaussian primes

$$Z + i = \pi_1^{k_1} \cdots \pi_r^{k_r}.$$

Taking norms and letting $N(\pi_i) = p_i$, we have

$$Z^2 + 1 = p_1^{k_1} \cdots p_r^{k_r}.$$

A Gaussian prime π_j satisfies exactly one of the following:

(i) $\pi_j = 1 + i$ or $\pi_j = 1 - i$ (in which case $\pi_j \mid 2$),

(ii) $N(\pi_j) \equiv 1 \bmod 4$, or

(iii) π_j is inert, and is a prime congruent to 3 mod 4 in $\mathbb{Z}$.

Therefore if Z^2+1 is divisible by a prime congruent to 3 mod 4, then there is an inert prime π_j dividing $Z+i$ and (by conjugation) also $Z-i$. But then π_j divides $Z+i-(Z-i)=2i$, which is impossible as π_j is inert and thus doesn't divide 2. Therefore Z^2+1 is not divisible by a prime congruent to 3 mod 4, and this concludes the proof of the sublemma. □

Returning to the proof of the theorem, it is a classical result that the number of ways of writing a non-square n as a sum of two squares is given by $\frac{1}{2}(e_1+1)(e_2+1)\cdots(e_r+1)$, where e_j is the multiplicity of the prime p_j congruent to 3 mod 4 dividing n. (This is under the assumption that n can be written as the sum of two squares, which is the case we are interested in). In this formulation, note that we consider X^2+Y^2 and $(-X)^2+Y^2$ to be the same representation. Therefore the number of ways of writing Z^2+1 is $\frac{1}{2}(k_1+1)(k_2+1)\cdots(k_r+1)$, excluding 2 and its exponent from the list. This is exactly half the number of odd divisors of Z^2+1, which we denote as $\frac{1}{2}d_o(Z^2+1)$.

As each individual representation $X^2+Y^2=Z^2+1$ comes with the eight lattice points $(\pm X, \pm Y, \pm Z)$, we see that the number of lattice point solutions to $X^2+Y^2=Z^2+1$ with $|Z| \leq R$ is given by

$$8\sum_{Z\leq R} \tfrac{1}{2}d_o(Z^2+1).$$

This completes the proof of the theorem. □

Note that when n is even, all the divisors of n^2+1 are odd and $d(n^2+1)=d_o(n^2+1)$. On the other hand, when n is odd, then n^2+1 is divisible by 2 exactly once, and $d(n^2+1)=2d_o(n^2+1)$. Therefore it is possible to convert the summation to $\sum d(n^2+1)$.

As a corollary to the proof of the above theorem, one can prove the following.

Corollary 6.0.3.

$$\begin{aligned}\#\{\textit{Integer points on } X^2+Y^2=(2Z)^2+1 \textit{ with } |Z|\leq R\} &= \sum_{n\leq R} 4d_o(4n^2+1)\\ &= \sum_{n\leq R} 4d(4n^2+1).\end{aligned}$$

While this hyperboloid is not studied in this thesis, the methodology still applies and it is possible to obtain asymptotics with error term for this lattice counting problem as well.

Further, if we denote

$$N_1(R) := \#\{\text{Integer points on } X^2 + Y^2 = Z^2 + 1 \text{ with } |Z| \leq R\}$$
$$N_2(R) := \#\{\text{Integer points on } X^2 + Y^2 = (2Z)^2 + 1 \text{ with } |Z| \leq R\},$$

then one can now easily compute that

$$\sum_{n \leq R} d(n^2 + 1) = \frac{N_1(R)}{2} - \frac{N_2(R/2)}{4}.$$

In this way, we convert a classical, and still somewhat mysterious, divisor sum into two lattice counting problems.

In the future, it would be a good idea to optimize the arguments in Chapter 5 in order to try to improve estimates for divisor sums. In particular, in the integral analysis for proving the main theorems, it is possible to further shift lines of integration and handle resulting residue terms (although one would also need to get a deeper understanding of the underlying analytic behavior).

Remark 6.0.4. Similar techniques may be applied to study

$$\sum_{n \leq R} d(n^2 + h)$$

for any positive h, although the computations look progressively messier.

Chapter Seven

Conclusion and Directions for Further Investigation

In the preceding, we considered two problems closely related to the classical Gauss Circle problem.

We first considered partial sums $S_f(n) = \sum_{m \leq n} a(n)$ of Fourier coefficients of a cusp form $f(z) = \sum a(n)e(nz)$, and showed that on average these partial sums behave like the error term in the Gauss Circle problem. To study these partial sums, we constructed the Dirichlet series

$$D(s, S_f \times S_g) = \sum_{n \geq 1} \frac{S_f(n)S_g(n)}{n^{s+k-1}},$$

as well as some related Dirichlet series, and showed that these Dirichlet series have meromorphic continuation to the plane.

The primary challenge comes from understanding the properties of the shifted convolution sum

$$\sum_{n,m \geq 1} \frac{a(n+m)a(n)}{(n+m)^s n^w}.$$

In Chapter 3, we approached this sum through a spectral expansion of a Poincare series appearing in an inner product against an appropriately chosen product of cuspforms. We showed that this spectral expansion is, to a large extent, explicitly understandable. By relating the properties of partial sums of coefficients of cusp forms to a particular spectral expansion, we explicitly relate the arithmetic properties of the coefficients to the very analytic properties of the spectrum of the hyperbolic Laplacian.

It is interesting to reflect on the successes and limitations of this approach. In Chapters 3 and 4, we showed that we are now able to prove many improvements to classical results. But we were unable to improve the estimate for the size of a single partial sum. Instead, we are only capable of matching the estimate due to Hafner and Ivić that

$$S_f(X) \ll X^{\frac{k-1}{2} + \frac{1}{3} + \epsilon}.$$

From the point of view of this thesis, the obstacle to further improvement was directly seen to be lack of understanding of the discrete spectrum (in terms of their distribution and cancellation) and the Lindelöf Hypothesis (or rather subconvexity) type bounds for Rankin-Selberg convolutions. Fundamentally, the techniques in this thesis are very different from previously attempted techniques — thus the similarity in the final bounds heuristically seems to represent an absolute bound on our current understanding.

In Chapter 4 we noted several other applications of the Dirichlet series $D(s, S_f \times S_g)$, including several projects that have already led to published papers [HKLDW17a, HKLDW17c, HKLDW16]. We have shown that the Dirichlet series $D(s, S_f \times S_g)$

present powerful tools to study a variety of questions related to the size, shape, and behavior of sums of coefficients of cusp forms. Through analysis of $D(s, S_f \times S_g)$, it is possible to prove results on long interval estimates, short interval estimates, and sign changes. More generally, one can apply many different Mellin integral transforms to $D(s, S_f \times S_g)$ in order to directly study different aspects.

One avenue of exploration that my collaborators and I have begun to explore it to perform an analogous construction of $D(s, S_f \times S_f)$ in cases when f is not a cusp form. This is proving to be a very interesting direction, and we will be able to explore more and different variants of the Gauss Circle problem. One particular direction is discussed at the very end of Chapter 4.

There is another interesting direction for further exploration, based on the techniques and observations from Chapters 3 and 4. As an individual Fourier coefficient is roughly of size $a(n) \sim n^{\frac{k-1}{2}}$ and the partial sum appears to satisfy $S_f(n) \ll n^{\frac{k-1}{2}+\frac{1}{4}}$, there is a large amount of cancellation among individual coefficients.

But I ask the following question: What if we consider partial sums formed from the partial sums $S_f(n)$? That is, what should we expect from the sizes of

$$\sum_{n \leq X} S_f(n)?$$

Further, what if we iterate this process and consider sums of sums of sums, and so on? Initial numerical investigation suggests that there continues to be extreme amounts of cancellation, far more than would occur by merely random chance.

In Chapter 5, we considered the question of how many points lie within the d-dimensional sphere of radius $\sqrt{R}$ and on the one-sheeted hyperboloid

$$\mathcal{H}_{h,d} = X_1^2 + \cdots + X_{d-1}^2 = X_d^2 + h,$$

which is essentially a constrained Gauss Sphere problem. We were able to prove improved bounds and asymptotics for this number of points.

On comparing the main techniques of Chapter 5 with those of Chapter 3, it is clear that there are many similarities. In both, we reduce the study towards sufficient understanding of a carefully chosen shifted convolution sum. And in both, we understand the convolution sum by translating the properties into properties of functions associated to the spectrum of the hyperbolic Laplacian.

The analysis and proof of the main theorem of Chapter 5 is not completely optimized. In particular, it is possible to perform a very close and detailed analytic argument, similar to the argument appearing in [HKLDW17c], in order to further

improve the bound on the main error term of the lattice point estimate. It may even be possible to improve estimates for the divisor sum $\sum d(n^2+1)$, whose connection with lattice points on hyperboloids is explained in Chapter 6.

More generally, the ideas and techniques of Chapter 5 can be extended to more general products of theta functions. By replacing the Jacobi theta function with theta functions associated to different quadratic forms, it should be possible to understand a wide variety of quadratic surfaces.

Chapter Eight

Original Motivations

Let me describe briefly what we set out to prove and how different the final results actually are.[†]

Chapters 3 and 4 were inspired from a single question that Jeff Hoffstein asked after a talk from Winfried Kohnen during Jeff's Birthday Conference. Kohnen described new results on sign changes of cusp forms, and Jeff asked whether or not it was possible to prove that sums of coefficients of cusp forms change sign frequently.

Very naïvely, I thought that if I could understand this question, I would probably use the series $D(s, S_f) = \sum_{n\geq 1} S_f(n)n^{-s}$ and $D(s, S_f \times \overline{S_f}) = \sum_{n\geq 1}|S_f(n)|^s n^{-s}$.

The reason for this is simple: one way to determine that a sequence changes signs often is to show that partial sums of squares are large while partial sums are small (indicating lots of cancellation). This fundamental idea was included within Kohnen's talk. As for the use of these Dirichlet series — it's in some sense the first thing that a multiplicative number theorist might try.

I doodled on this question for a few talks, and shared my doodles with Tom Hulse. Tom had two quick ideas: firstly, he knew of an exact set of conditions that guarantee sign changes in intervals; secondly, he knew a Mellin-Barnes transform to decouple denominators. This seemed like a short, quick project, so we set out to prove sign changes.

We were wrong. This was not short nor quick, and as we delved into the problem it became apparent that there were significant obstacles in the way of understanding the spectral analysis. In some sense, we knew that these were understandable from the general philosophy of [HHR13]. Yet actually demonstrating the extent of cancellation required attention to different subtleties. And I engrossed myself into these details.

Several months later, the question had totally shifted away from sign changes, and instead focused on the Cusp Form analogy to the Gauss Circle problem in various aspects, leading ultimately to the current line of research. In hindsight, it turns out that our analysis of $D(s, S_f)$ and $D(s, S_f \times \overline{S_f})$ have had only limited success in actually proving sign changes — but they are great tools otherwise.

Chapters 5 and 6 accomplished almost exactly what was originally intended. In [HKKrL16], Hulse, Kıral, Kuan, and Lim studied a problem inspired from the same work of Oh and Shah [OS14] that studied lattice points on hyperboloids through ergodic methods. Shortly afterwards, a Kıral, Kuan, and I began to look at lattice points on hyperboloids. Our initial investigations stalled, though I first learned

[†]Jeff Hoffstein once told me that far too few published works describe the difference between the original intent and the final version. I include that here, in this very informal section.

much about the general techniques involving shifted convolution sums from those first forays into this problem.

It is interesting to note that the combination of integral transforms that leads to the main theorem of Chapter 5 was noticed by a combination of Kuan, Walker, and I originally while we were writing our sign changes paper [HKLDW16]. We then forgot about this technique until we began to struggle with our forthcoming work on the Gauss Sphere problem, until Walker and I slowly realized that we were vastly overcomplicating a particular difficulty.

Originally, the intention of Chapter 5 was simply to prove the meromorphic continuation of the underlying Dirichlet series and to prove the smoothed sum result. This is a rare case of proving exactly what I had set out to prove, and then a little bit more.[‡]

[‡]As opposed to the normal pattern of failing in the original goal, and then proving something else entirely. As Jeff likes to say, it is important to be able to love the theorem that you can prove, as rarely can we prove the theorems that we love.

Index

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